How to Tee a Hyperplane

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Abstract. A letter tee (T) is a geometric figure consisting of two line segments meeting at a right angle. We construct a covering of a hyperplane by a disjoint family of tees.

1. INTRODUCTION. Fix a positive integer n. We identify $\mathbb{R}^n$ with the set of elements in $\mathbb{R}^{n+1}$ whose last coordinate is 0, and we set

$$\mathbb{R}^{n+1}_+ := \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} \geq 0\}.$$

A *tee* at a point $p \in \mathbb{R}^n$ is a union of two line segments in $\mathbb{R}^{n+1}_+$: the first segment (the *base*) has p as an endpoint and is orthogonal to $\mathbb{R}^n$, and the second segment (the *top*) is bisected by the other endpoint of the base and is perpendicular to the base (and therefore is parallel to $\mathbb{R}^n$); see Figure 1. To *tee* $\mathbb{R}^n$ is to place a tee at every point $p \in \mathbb{R}^n$ in such a way that no two tees intersect.

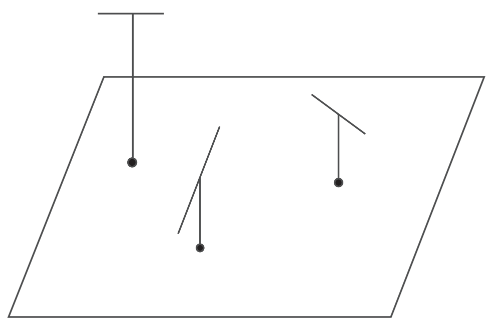

Figure 1. The plane $\mathbb{R}^2 \subset \mathbb{R}^3$ with three tees of different sizes and orientations.

Ben Schmidt [7] asks the following question:

Question. *For which n it is possible to tee* $\mathbb{R}^n$*?*

The purpose of this note is to answer Schmidt's question.

Theorem 1. *It is possible to tee* $\mathbb{R}^n$ *if and only if* $n \geq 2$.

Many other surprising coverings of Euclidean space are known. Here we mention two such coverings.

- Consider a set S and a positive integer k. A collection $\{S_i : i \in I\}$ of subsets of S is called a *k-homogeneous covering* if, for every $s \in S$, the set $\{i \in I : s \in S_i\}$ has exactly k elements. Kharazishvili and Tetunashvili [3] show that for every $k \geq 2$, there is a k-homogeneous covering of $\mathbb{R}^2$ consisting of circles of diameter 1 (for $k = 1$, a covering is impossible even if the diameters of the circles are not fixed). It is also possible to construct a 1-homogeneous covering of $\mathbb{R}^3$ by circles of diameter 1.

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- A *spray* centered at a point $x \in \mathbb{R}^2$ is a set that has finite intersection with each circle centered at x. Schmerl [6] proves that given three points $c_1, c_2, c_3 \in \mathbb{R}^2$ not on a line, there exist sprays S_1, S_2, S_3 centered at c_1, c_2, c_3, respectively, that cover $\mathbb{R}^2$. Surprisingly, the corresponding statement for c_1, c_2, c_3 collinear is equivalent to the continuum hypothesis [1].

There are many other examples of results of similar flavor. We refer the interested reader to the survey [4].

2. THE PROOF. The real projective space $\mathbb{RP}^{n-1}$ is the space of lines through the origin in $\mathbb{R}^n$. For each $p \in \mathbb{R}^n$, the space of lines through p has a natural identification with $\mathbb{RP}^{n-1}$. A tee-ing of $\mathbb{R}^n$ can be described by a *tee function*

$$f : \mathbb{R}^n \to \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{RP}^{n-1}$$

that takes $p \in \mathbb{R}^n$ to the length of the base, length of the top, and the projection to $\mathbb{R}^n$ of the top of the tee at p. Schmidt [7] proved that a tee function cannot be continuous on any nonempty open set. Our proof of Theorem 1 is nonconstructive, and we expect every teeing must be wild.

More specifically, the proof of Theorem 1 uses a well-ordering of $\mathbb{R}^n$. The well-ordering theorem, which is equivalent to the axiom of choice, says that every set can be equipped with a well-ordering, and it is consistent with the axioms of Zermelo–Fraenkel set theory (which exclude the axiom of choice) that $\mathbb{R}^n$ cannot be well-ordered. The proof of our Theorem 1 also uses Zorn's lemma, which itself is equivalent to the axiom of choice.

Proof of Theorem 1. The case $n = 1$ is an old problem, and in this case every collection of disjoint tees must be countable. Moreover, a result of Moore [5] from the 1920s implies that the result is still true if we only require the tees to be continuously embedded (i.e., if we do not require the base and top to be straight).

For the sake of completeness, we give an argument for the $n = 1$ case. Let $S \subset \mathbb{R}$ and suppose we have chosen a tee at each point of S such that the tees are disjoint. We will show that S is countable. For each $p \in S$, pick rational numbers $a_p < b_p$ such that the interval (a_p, b_p) contains p and is contained in the projection of the top of the tee at p onto $\mathbb{R}$.[1] Consider the map $S \to \mathbb{Q}^2$, taking $p \in S$ to $(a_p, b_p) \in \mathbb{Q}^2$.

This map must be injective, for if $p \neq q$ but $(a_p, b_p) = (a_q, b_q)$, then among the tees at p and q, the base of one must intersect the top of the other. This proves the claim, as $\mathbb{Q}^2$ is countable.

Now suppose $n \geq 2$. The well-ordering theorem implies there exists a well-ordering on $\mathbb{R}^n$. We choose a well-ordering $\preceq$ such that the order type of $(\mathbb{R}^n, \preceq)$ is minimal (see [2, Theorem 1.3, p. 130]). We say a subset $S \subset \mathbb{R}^n$ is an *initial segment* if $p \in S$ and $q \preceq p$ imply $q \in S$. The condition on the order type of $(\mathbb{R}^n, \preceq)$ implies that every proper initial segment has cardinality strictly smaller than the continuum.

Let $\mathcal{C}$ be the set of pairs $(S, \{(h_p, \ell_p) : p \in S\})$ that satisfy all of the following conditions:

1. $S \subset \mathbb{R}^n$ is an initial segment.
2. For each $p \in S$, h_p is a positive real number and $\ell_p \subset \mathbb{R}^n$ is a line through p.
3. If $p \prec q \in S$, then $p \notin \ell_q$ and $h_p \neq h_q$.
4. If $p \prec q \in S$ and $q \in \ell_p$, then $h_p > h_q$.
5. For every $q \in \mathbb{R}^n$, the set $\{p \in S \backslash \{q\} : q \in \ell_p\}$ has at most two elements.

[1] Among the possible choices for a_p and b_p, we can choose those with minimal denominator, breaking ties by minimizing the numerator. So this step does not involve the axiom of choice.

The set $\mathcal{C}$ is partially ordered by inclusion in both coordinates, and the coordinatewise union of a totally ordered subset of $\mathcal{C}$ is again in $\mathcal{C}$, so Zorn's lemma implies there is a maximal element

$$(S, \{(h_p, \ell_p) : p \in S\}) \in \mathcal{C}. \tag{1}$$

We claim that this maximal element necessarily satisfies $S = \mathbb{R}^n$.

For the sake of contradiction, assume $S \subsetneq \mathbb{R}^n$. Let p_0 be the minimal element of $\mathbb{R}^n \backslash S$ (with respect to the chosen well-ordering). We will find $h_{p_0} > 0$ and a line ℓ_{p_0} such that $(S \cup \{p_0\}, \{(h_p, \ell_p)\})$ satisfies conditions (1)–(5), contradicting the maximality of (1). To satisfy condition (3), we need ℓ_{p_0} not to intersect S, so this excludes $\#S$ possibilities for ℓ_{p_0}. Additionally, to satisfy condition (5), we need ℓ_{p_0} not to contain the point of intersection of ℓ_p and ℓ_q for any $p \neq q \in S$, and since condition (3) implies $\ell_p \neq \ell_q$ for $p \neq q$, this excludes at most $\#(S \times S)$ possibilities for ℓ_{p_0}. Because S is a proper initial segment, S and $S \times S$ have cardinality strictly smaller than the continuum. The set of lines passing through p_0 has continuum cardinality, so we can find a line ℓ_{p_0} through p_0 satisfying the desired property. Next, the set $\{h_p : p \in S, p_0 \in \ell_p\}$ has at most two elements, and we choose $h_{p_0} > 0$ to be smaller than the minimum value of this set (this takes care of condition (4)), and to exclude the $\#S$ values $\{h_p : p \in S\}$ (this takes care of condition (3)). Then $(S \cup \{p_0\}, \{(h_p, \ell_p)\})$ satisfies conditions (1)–(5) above, contradicting the maximality of $(S, \{(h_p, \ell_p)\})$. This proves the claim.

Now we are ready to construct the teeing. Let $(\mathbb{R}^n, \{(h_p, \ell_p)\})$ be a maximal element of $\mathcal{C}$. For each $p \in \mathbb{R}^n$, let T_p be a tee at p whose height is h_p, such that the projection of the top of T_p to $\mathbb{R}^n$ is contained in ℓ_p (the width of the top of T_p is immaterial and can be chosen arbitrarily). For $p \prec q \in \mathbb{R}^n$, condition (3) above implies T_p does not intersect the top of T_q, and condition (4) implies T_p does not intersect the base of T_q. ■

3. GENERALIZATIONS. It is natural to ask for generalizations to other letters of the alphabet. We invite the reader to adapt the proof above to the following situations.

- A *vee* (V) at a point $p \in \mathbb{R}^n$ is the union of two distinct line segments in $\mathbb{R}^{n+1}_+$ of the same length, each of which has p as an endpoint, such that the line bisecting the angle formed by those segments is orthogonal to $\mathbb{R}^n$.

  Exercise. Prove it is possible to vee $\mathbb{R}^n$ if and only if $n \geq 2$.

- An *o* (O) at a point $p \in \mathbb{R}^n$ is a circle in $\mathbb{R}^{n+1}_+$ containing p, such that the radius of the circle through p is orthogonal to $\mathbb{R}^n$.

  Exercise. Prove it is possible to o $\mathbb{R}^n$ if and only if $n \geq 2$.

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A Short Proof for the Volume of Square Pyramids

Let $V(a, h)$ denote the volume of a square pyramid P with height h and base $a \times a$. The pictures below show that P can be decomposed into a parallelepiped and two smaller pyramids with half the dimensions of P, that is

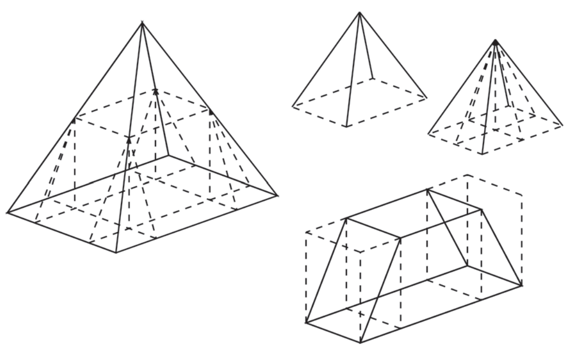
$$V(a, h) = a^2h/4 + 2V(a/2, h/2).$$

Repeating the decomposition process for the smaller pyramids and so on, we get

$$V(a, h) = \frac{a^2h}{4} + \frac{a^2h}{4}\left(\frac{1}{4}\right) + \cdots + \frac{a^2h}{4}\left(\frac{1}{4}\right)^{k-1} + 2^k V(a/2^k, h/2^k),$$

$k \geq 1$. Because $2^k V(a/2^k, h/2^k) \leq 2^k \left(\frac{a}{2^k}\right)^2 \left(\frac{h}{2^k}\right)$, the remainder term converges to zero. Thus,

$$V(a, h) = \frac{a^2h}{4}\left(1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots\right) = \frac{a^2h}{3}.$$



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