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The Completed Finite Period Map and Galois Theory of Supercongruences

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A period is a complex number arising as the integral of a rational function with algebraic number coefficients over a region cut out by finitely many inequalities between polynomials with rational coefficients. Although periods are typically transcendental numbers, there is a conjectural Galois theory of periods coming from the theory of motives. This paper formalizes an analogy between a class of periods called multiple zeta values and congruences for rational numbers modulo prime powers (called supercongruences). We construct an analog of the motivic period map in the setting of supercongruences and use it to define a Galois theory of supercongruences. We describe an algorithm using our period map to find and prove supercongruences, and we provide software implementing the algorithm.

1 Introduction

1.1 Periods

A *period* is a complex number given by the integral of a rational function with algebraic number coefficients, over a region in $\mathbb{R}^n$ defined by finitely many inequalities between polynomials with rational coefficients. Many familiar constants are periods,

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for example,

$$\pi = \iint_{x^2+y^2\leq 1} 1\,dx\,dy, \quad \log(r) = \int_1^r \frac{dx}{x} \text{ for } r \in \mathbb{Q}_{>0},$$

$$\zeta(n) = \int_0^1 \cdots \int_0^1 \frac{dx_1 \ldots dx_n}{1 - x_1 \cdots x_n} \text{ for } n \geq 2.$$

The set $\mathcal{P} \subset \mathbb{C}$ of all periods is a countable subring of $\mathbb{C}$ containing the algebraic numbers. Although periods are typically transcendental numbers, the theory of motives predicts that a version of Galois theory should hold for periods (see [1]). The motivic Galois action has been studied in depth for a class of periods called multiple zeta values [4], which we now define.

Recall that a *composition* is a finite ordered list $\underline{\mathbf{s}} = (s_1, \ldots, s_k)$ of positive integers. The *weight* and *depth* of $\underline{\mathbf{s}}$ are $|\underline{\mathbf{s}}| = s_1 + \ldots + s_k$ and $\ell(\underline{\mathbf{s}}) = k$, respectively. For $\underline{\mathbf{s}}$ a composition satisfying $s_1 \geq 2$, we define the *multiple zeta value* by the convergent infinite series

$$\zeta(\underline{\mathbf{s}}) := \sum_{n_1 > \ldots > n_k \geq 1} \frac{1}{n_1^{s_1} \ldots n_k^{s_k}} \in \mathbb{R}. \tag{1.1}$$

For every composition $\underline{\mathbf{s}}$, we have

$$\zeta(\underline{\mathbf{s}}) = \int \cdots \int_{0 \leq x_1 \leq \ldots \leq x_{|\underline{\mathbf{s}}|} \leq 1} \omega_1 \ldots \omega_{|\underline{\mathbf{s}}|},$$

where $\omega_i = dx_i/(1 - x_i)$ if $i \in \{s_1, s_1 + s_2, \ldots, s_1 + \ldots + s_k\}$ and $\omega_i = dx_i/x_i$ otherwise. This iterated integral expression shows that multiple zeta values are periods.

1.2 Supercongruences

This paper develops a connection between periods and prime power divisibility properties of sequences of rational numbers. A *supercongruence* is a congruence between rational or p-adic numbers modulo a power of a prime p. We consider families of supercongruences holding for all primes at once, up to finitely many exceptions. For example, in 1979 Apéry [2] proved that $\zeta(3)$ is irrational. The proof involved a sequence of rational approximations to $\zeta(3)$, whose denominators are given by the integers

$$a_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2,$$

now called the *Apéry numbers*. It is known [8] that the Apéry numbers satisfy the supercongruence $a_p \equiv 5 \bmod p^3$ for every prime $p \geq 5$. In our setting, we view the prime-indexed sequence (a_p) as a finite analog of a period.

Finite truncations of the multiple zeta value series (1.1) are called *multiple harmonic sums*, and we write

$$H_N(s_1, \ldots, s_k) := \sum_{N \geq n_1 > \ldots > n_k \geq 1} \frac{1}{n_1^{s_1} \ldots n_k^{s_k}} \in \mathbb{Q}.$$

Many supercongruences are known for multiple harmonic sums, especially in the case $N = p - 1$, with p a prime. Residue classes of multiple harmonic sums $H_{p-1}(\underline{\mathbf{s}})$ modulo p (or sometimes modulo powers of p) are called finite multiple zeta values, and they have received considerable attention in recent years (e.g., see the recent works [15, 19–22, 25, 26, 31]).

A useful technique for proving supercongruences is to relate terms to multiple harmonic sums. For example, consider the central binomial coefficient $\binom{2p}{p}$, which has interesting arithmetic properties. It is not difficult to show that

$$\binom{2p}{p} = 2 \sum_{n=0}^{\infty} p^n H_{p-1}(\underbrace{1, \ldots, 1}_{n}). \tag{1.2}$$

Many series expansions related to (1.2) are given by the author in [24].

1.3 *p*-adic multiple zeta values

The bridge between periods and supercongruences comes from a p-adic analog $\zeta_p(\underline{\mathbf{s}}) \in \mathbb{Q}_p$ of the multiple zeta values. These p-adic numbers record the action of the crystalline Frobenius on the motivic unipotent fundamental group of $\mathbb{P}^1 \backslash \{0, 1, \infty\}$ ([10], §5.28). There is a construction in terms of the Coleman integral due to Furusho [11, 12]. The p-adic multiple zeta values are expected to satisfy the same algebraic relations as the real multiple zeta values, along with the additional relation $\zeta_p(2) = 0$.

A remarkable formula (stated as Theorem 2.1 below), conjectured by Hirose and Yasuda [29] and independently discovered and proved by Jarossay [18] expresses the multiple harmonic sum $H_{p-1}(\underline{\mathbf{s}})$ as a p-adically convergent infinite linear combination of p-adic multiple zeta values. As a consequence, any p-adically convergent series involving multiple harmonic sums $H_{p-1}(\underline{\mathbf{s}})$ can be rewritten as series involving p-adic multiple zeta values. For example, (1.2) can be used to derive a p-adically convergent series representation

$$\binom{2p}{p} = 2 - 4\zeta_p(3) - 12\zeta_p(5) + 4\zeta_p(3)^2 - 36\zeta_p(7) + \dots \tag{1.3}$$

in terms of p-adic multiple zeta values.

1.4 Motivic periods

The Galois theory of periods is conditional on algebraic independence statements for periods. To obtain an unconditional theory, one replaces $\mathcal{P}$ by an abstractly defined ring $\mathcal{P}^{\mathfrak{m}}$, called a ring of formal (or motivic) periods. One also defines a linear pro-algebraic group G and an action of G on $\mathcal{P}^{\mathfrak{m}}$. The motivic *period map* is then a ring homomorphism $per : \mathcal{P}^{\mathfrak{m}} \to \mathbb{C}$ taking a motivic period to an actual (complex) period. The Grothendieck period conjecture for $\mathcal{P}^{\mathfrak{m}}$ is the statement that per is injective, and if this is the case one gets an action of G on $\mathcal{P}$ (see [14], Chapter 12).

The multiple zeta values are the periods of mixed Tate motives over $\mathbb{Z}$ [4]. Here the corresponding motivic period ring is $\mathcal{H}$, the *ring of motivic multiple zeta values*. This is a commutative $\mathbb{Q}$-algebra spanned by elements $\zeta^{\mathfrak{m}}(\underline{\mathbf{s}})$, and the period map $per\colon \mathcal{H} \to \mathbb{R}$ takes $\zeta^{\mathfrak{m}}(\underline{\mathbf{s}})$ to $\zeta(\underline{\mathbf{s}})$. For p-adic multiple zeta values, the motivic period ring is $\mathcal{P}^{\mathfrak{dr}}$, the ring of de Rham periods of the category of mixed Tate motives over $\mathbb{Z}$ (see [6], §2.8). There is an invertible element $\mathbb{L} \in \mathcal{P}^{\mathfrak{dr}}$ called the de Rham Lefschetz period, and $\mathcal{P}^{\mathfrak{dr}}$ is spanned as a $\mathbb{Q}[\mathbb{L}, \mathbb{L}^{-1}]$-module by elements $\zeta^{\mathfrak{dr}}(\underline{\mathbf{s}})$ as $\underline{\mathbf{s}}$ ranges over compositions. For each prime p, the crystalline Frobenius on mixed Tate motives induces a ring homomorphism $per_p : \mathcal{P}^{\mathfrak{dr}} \to \mathbb{Q}_p$, taking $\zeta^{\mathfrak{dr}}(\underline{\mathbf{s}})$ to $\zeta_p(\underline{\mathbf{s}})$ and taking $\mathbb{L}$ to p.

1.5 Results

1.5.1 *The completed finite period map*

We construct an analog $\widehat{per}$ of the period maps per and per_p in the setting of finite periods. We need to consider infinite linear combinations of p-adic multiple zeta values (e.g., (1.3)), so the domain of $\widehat{per}$ is the completion $\widehat{\mathcal{P}}^{\mathfrak{dr}}$ of $\mathcal{P}^{\mathfrak{dr}}$ with respect to the weight grading (where $\zeta^{\mathfrak{dr}}(\underline{\mathbf{s}})$ has weight $|\underline{\mathbf{s}}|$ and $\mathbb{L}$ has weight 1). The codomain of $\widehat{per}$ is the quotient ring

$$\mathbb{Q}_{p\to\infty} := \frac{\left\{(a_p) \in \prod_p \mathbb{Q}_p : v_p(a_p) \text{ bounded below}\right\}}{\left\{(a_p) \in \prod_p \mathbb{Q}_p : v_p(a_p) \to \infty \text{ as } p \to \infty\right\}}.$$

The rings $\widehat{\mathcal{P}}^{\mathfrak{dr}}$ and $\mathbb{Q}_{p\to\infty}$ are complete with respect to topologies arising from decreasing filtrations. The filtration on $\widehat{\mathcal{P}}^{\mathfrak{dr}}$ comes from the weight grading, and the filtration on $\mathbb{Q}_{p\to\infty}$ is given by

$$\mathrm{Fil}^n \mathbb{Q}_{p\to\infty} := \{(a_p) \in \mathbb{Q}_{p\to\infty} : \liminf v_p(a_p) \geq n\}.$$

Equation (2.2) below implies $v_p(\zeta_p(\underline{\mathbf{s}})) \geq |\underline{\mathbf{s}}|$ for $p > |\underline{\mathbf{s}}|$, so that $(\zeta_p(\underline{\mathbf{s}}))$ is an element of $\mathrm{Fil}^{|\underline{\mathbf{s}}|}\mathbb{Q}_{p\to\infty}$.

Definition 1.1. The *completed finite period map* is the continuous ring homomorphism

$$\widehat{per} : \widehat{\mathcal{P}}^{\mathfrak{dr}} \to \mathbb{Q}_{p\to\infty}$$

induced by the maps per_p. It is the unique continuous ring homomorphism satisfying $\widehat{per}(\zeta^{\mathfrak{dr}}(\underline{\mathbf{s}})) = (\zeta_p(\underline{\mathbf{s}}))$ and $\widehat{per}(\mathbb{L}) = (p)$.

In [24] the author introduces a subalgebra of $\mathbb{Q}_{p\to\infty}$ called the MHS algebra, consisting of elements that admit p-adic series expansion of a certain shape involving multiple harmonic sums, generalizing (1.2). A precise definition of the MHS algebra is given below as Definition 3.1. The MHS algebra contains many "elementary" quantities (various sums of binomial coefficients, generalizations of the harmonic numbers, etc.), some of which are listed in Theorem 3.2 below.

The following result is Theorem 3.3 below.

Theorem 1.2. The image of $\widehat{per}$ is precisely the MHS algebra.

We also formulate an analog of the period conjecture, which says that $\widehat{per}$ is compatible with the filtrations on its domain and codomain in the following strong sense.

Conjecture 1.3. (Period conjecture) The period map $\widehat{per}$ satisfies

$$\widehat{per}^{-1}\left(\mathrm{Fil}^n \mathbb{Q}_{p\to\infty}\right) = \mathrm{Fil}^n\, \widehat{\mathcal{P}}^{\mathfrak{dr}}$$

for all n. In particular, $\widehat{per}$ is injective.

The truth of Conjecture 1.3 would give a completely algorithmic way to prove supercongruences between elements of the MHS algebra. We describe the algorithm in Section 4. We also provide software implementing the algorithm, which we describe in Appendix A.

1.5.2 Galois theory

There is a pro-algebraic group G_{dR} that acts faithfully on $\mathcal{P}^{\mathfrak{dr}}$. The action respects all algebraic relations holding in $\mathcal{P}^{\mathfrak{dr}}$, and the group G_{dR} is very large. The truth of Conjecture 1.3 would imply that the action of G_{dR} descends to the MHS algebra, in which case we would get a Galois theory of supercongruences. We describe this Galois theory of supercongruences in Section 5.

The result of the action of G_{dR} on an element of the MHS algebra is defined as an infinite series involving p-adic multiple zeta values and is a priori difficult to compute. However, in some cases, the result of the action has an elementary description in terms of sequences of rational numbers. We compute some examples of this in Section 6, and we use our computations to give an (unconditional) proof of a supercongruence for factorials.

2 The completed finite period map

In this section we construct the completed finite period map.

2.1 p-adic multiple zeta values

The unipotent fundamental groupoid of $X = \mathbb{P}^1 \backslash \{0, 1, \infty\}$ has the structure of a motivic groupoid [9]. This means that for each $x, y \in X(\mathbb{Q})$, there are various pro-algebraic varieties $\pi_{1,\bullet}(X;x, y)$, called *realizations*, corresponding to different cohomology theories. We will make use of the de Rham, Betti, and crystalline realizations:

$$\pi_{1,dR}(X;x,y), \qquad \pi_{1,B}(X;x,y), \qquad \pi_{1,p}(X;x,y). \tag{2.1}$$

Through the use of tangential base points, (2.1) makes sense when x and y are tangent vectors at 0 or 1. We consider unit tangent vectors t_{01} at 0 in the positive direction and t_{10} at 1 in the negative direction, and we write $\Pi_{01,\bullet}$ for $\pi_{1,\bullet}(X;t_{01}, t_{10})$, where $\bullet \in \{dR, B, p\}$.

For any commutative $\mathbb{Q}$-algebra R, we may identify the R-points of $\Pi_{01,dR}$ with the set of group-like elements of $R \ll e_0, e_1 \gg$, the complete Hopf algebra of formal power series in non-commuting variables e_0 and e_1. The straight line path from 0 to 1 determines an element $dch \in \Pi_{01,B}(\mathbb{Q})$, and under the Betti–de Rham comparison isomorphism

$$C_{B,dR} : \Pi_{01,B} \times \operatorname{Spec} \mathbb{C} \xrightarrow{\sim} \Pi_{01,dR} \times \operatorname{Spec} \mathbb{C},$$

dch maps to an element of $\Pi_{01,dR}(\mathbb{C})$. In the power series associated with $C_{B,dR}(dch)$, the coefficient of

$$e_0^{s_1-1} e_1 \ldots e_0^{s_k-1} e_1$$

is $(-1)^k \zeta(s_1, \ldots, s_k)$.

Every vector bundle on X with unipotent connection has a canonical trivialization, which determines an element $\gamma \in \Pi_{01,dR}(\mathbb{Q})$. Via the de Rham–crystalline comparison isomorphism, the crystalline Frobenius gives an automorphism ϕ_p of

$$\Pi_{01,p} \simeq \Pi_{01,dR} \times \operatorname{Spec} \mathbb{Q}_p.$$

The image of γ under ϕ_p is an element of $\Pi_{01,dR}(\mathbb{Q}_p)$. For each composition $\underline{\mathbf{s}} = (s_1, \ldots, s_k)$, the p-adic multiple zeta value $\zeta_p(\underline{\mathbf{s}})$ is defined to be $(-1)^k$ times the coefficient of

$$e_0^{s_1-1} e_1 \ldots e_0^{s_k-1} e_1$$

in $\phi_p(\gamma)$ (see [18], §2.4.1).

It is known that

$$\zeta_p(\underline{\mathbf{s}}) \in \sum_{\ell \geq |\underline{\mathbf{s}}|} \frac{p^\ell}{\ell!} \mathbb{Z}_p. \tag{2.2}$$

This fact is stated in [7] (Corollary 5.5) for a different version of the p-adic multiple zeta values, but the relationship between the p-adic multiple zeta values of [7] and those considered here is well-known, and the result also holds in our setting. In particular, $\zeta_p(\underline{\mathbf{s}}) \in p^{|\underline{\mathbf{s}}|}\mathbb{Z}_p$ whenever $p > |\underline{\mathbf{s}}|$.

The following formula expresses the multiple harmonic sum $H_{p-1}(\underline{\mathbf{s}})$ as an infinite series involving p-adic multiple zeta values.

Theorem 2.1. ([18], Corollary 2). For every composition $\underline{\mathbf{s}} = (s_1, \ldots, s_k)$, there is a p-adically convergent infinite series identity

$$p^{|\underline{\mathbf{s}}|} H_{p-1}(\underline{\mathbf{s}}) = \sum_{i=0}^{k} \sum_{\ell_1,\ldots,\ell_i \geq 0} (-1)^{s_1+\ldots+s_i} \prod_{j=1}^{i} \binom{s_j+\ell_j-1}{\ell_j} \zeta_p(s_i+\ell_i, \ldots, s_1+\ell_1) \zeta_p(s_{i+1}, \ldots, s_k). \tag{2.3}$$

By (2.2), the convergence of the infinite series on the right hand side of (2.3) is uniform in p in the following sense. For every positive integer N, the congruence

$$p^{|\underline{\mathbf{s}}|} H_{p-1}(\underline{\mathbf{s}}) \equiv \sum_{i=0}^{k} \sum_{\substack{\ell_1,\ldots,\ell_i \geq 0 \\ \ell_1+\ldots+\ell_i < N}} (-1)^{s_1+\ldots+s_i} \prod_{j=1}^{i} \binom{s_j+\ell_j-1}{\ell_j}$$
$$\zeta_p(s_i+\ell_i, \ldots, s_1+\ell_1) \zeta_p(s_{i+1}, \ldots, s_k) \mod p^{|\underline{\mathbf{s}}|+N}$$

holds for all but finitely many p. The truth of Conjecture 2.3 below would imply that this uniformity condition uniquely characterizes the right hand side of (2.3), up to motivic relations among the ζ_p.

2.2 The target ring

In [23], the author defined a finite analog of the multiple zeta function. The codomain of this map was a complete topological ring related to the finite adeles. Here we use this ring with p inverted. Adopting the notation of [17], we define

$$\mathbb{Q}_{p\to\infty} := \frac{\left\{(a_p) \in \prod_p \mathbb{Q}_p : v_p(a_p) \text{ bounded below}\right\}}{\left\{(a_p) \in \prod_p \mathbb{Q}_p : v_p(a_p) \to \infty \text{ as } p \to \infty\right\}}.$$

We equip $\mathbb{Q}_{p\to\infty}$ with a decreasing, exhaustive, separated filtration:

$$\mathrm{Fil}^n \mathbb{Q}_{p\to\infty} = \left\{(a_p) \in \mathbb{Q}_{p\to\infty} : \liminf v_p(a_p) \geq n\right\}.$$

The subsets $\mathrm{Fil}^n \mathbb{Q}_{p\to\infty} \subset \mathbb{Q}_{p\to\infty}$ form a neighborhood basis of 0 for a topology, and $\mathbb{Q}_{p\to\infty}$ is complete because it is the quotient of the complete, first countable ring $\prod_p \mathbb{Q}_p$ with the uniform topology (i.e., the sets $\prod_p p^n \mathbb{Z}_p$ are a neighborhood basis of 0) modulo a closed ideal.

Convergence in $\mathbb{Q}_{p\to\infty}$ is distinct from p-adic convergence. A sequence $(a_{p,1})$, $(a_{p,2})$, ... converges to $(b_p) \in \mathbb{Q}_{p\to\infty}$ if and only if for every positive integer m, there exists $N = N(m)$ such that for all $n > N$ the congruence

$$a_{p,n} \equiv b_p \mod p^m\mathbb{Z}_p \tag{2.4}$$

holds for all but finitely many p. The finite set of primes for which (2.4) fails may depend on n, so convergence in $\mathbb{Q}_{p\to\infty}$ does *not* imply that $a_{p,n}$ converges p-adically to b_p for any p at all. An example is

$$a_{p,n} = \begin{cases} 1 : p \le n, \\ 0 : p > n. \end{cases}$$

Then $a_{p,n} \to 1$ for every p, but $(a_{p,n}) \to 0$ in $\mathbb{Q}_{p\to\infty}$ (and in fact $(a_{p,n})$ is the constant sequence 0).

The ring $\mathbb{Q}_{p\to\infty}$ is non-Archimedean, so an infinite series converges if and only if the terms go to 0 (i.e., for every integer n, all but finite many terms in the series are in $\mathrm{Fil}^n\mathbb{Q}_{p\to\infty}$). Concretely, if $(a_{p,n}) \to 0$, we have $\sum_n (a_{p,n}) = (a_p)$, where

$$a_p = \sum_{\substack{n \\ v_p(a_{p,n}) \ge \liminf_\ell v_\ell(a_{\ell,n})}} a_{p,n}.$$

2.3 The period map

The category $MT(\mathbb{Z})$ of mixed Tate motives unramified over $\mathbb{Z}$ is a neutral Tannakian category over $\mathbb{Q}$. The de Rham realization of the fundamental group of $MT(\mathbb{Z})$ is an affine pro-algebraic group over $\mathbb{Q}$, which we denote G_{dR}. The *ring of de Rham periods* $\mathcal{P}^{\mathfrak{dr}}$ is defined to be the coordinate ring of G_{dR}, which is a commutative Hopf algebra over $\mathbb{Q}$. The de Rham Lefschetz period is an element $\mathbb{L} \in (\mathcal{P}^{\mathfrak{dr}})^\times$. For each composition $\underline{\mathbf{s}}$, there is a *de Rham multiple zeta value* $\zeta^{\mathfrak{dr}}(\underline{\mathbf{s}}) \in \mathcal{P}^{\mathfrak{dr}}$, and $\mathcal{P}^{\mathfrak{dr}}$ is spanned as a $\mathbb{Q}[\mathbb{L}, \mathbb{L}^{-1}]$-module by the de Rham multiple zeta values. The elements $\zeta^{\mathfrak{dr}}(\underline{\mathbf{s}})$ satisfy the same relations as the motivic multiple zeta values $\zeta^{\mathfrak{m}}(\underline{\mathbf{s}})$, along with the additional relation $\zeta^{\mathfrak{dr}}(2) = 0$.

There is a grading on $\mathcal{P}^{\mathfrak{dr}}$ by weight, where $\mathbb{L}$ has weight 1 and $\zeta^{\mathfrak{dr}}(\underline{\mathbf{s}})$ has weight $|\underline{\mathbf{s}}|$. The domain of our period map is the completion $\widehat{\mathcal{P}}^{\mathfrak{dr}}$ of $\mathcal{P}^{\mathfrak{dr}}$ with respect to the weight grading. An element of $\widehat{\mathcal{P}}^{\mathfrak{dr}}$ may be viewed as a formal infinite sum

$$\sum_{i=0}^{\infty} a_i \mathbb{L}^{b_i} \zeta^{\mathfrak{dr}}(\underline{\mathbf{s}}_i), \tag{2.5}$$

where $a_i \in \mathbb{Q}$, $b_i \in \mathbb{Z}$, and $\underline{\mathbf{s}}_i$ are compositions satisfying $b_i + |\underline{\mathbf{s}}_i| \to \infty$ as $i \to \infty$. We write $\mathrm{Fil}^\bullet$ for the weight filtration on $\widehat{\mathcal{P}}^{\mathfrak{dr}}$, where $\mathrm{Fil}^n\widehat{\mathcal{P}}^{\mathfrak{dr}}$ consists of those elements (2.5) such that $b_i + |\underline{\mathbf{s}}_i| \geq n$ whenever $a_i \neq 0$.

The integrality result (2.2) implies that $\zeta_p(\underline{\mathbf{s}}) \in p^{|\underline{\mathbf{s}}|}\mathbb{Z}_p$ for $p > |\underline{\mathbf{s}}|$, so that $(p^b\zeta_p(\underline{\mathbf{s}})) \in \mathrm{Fil}^{b+|\underline{\mathbf{s}}|}\mathbb{Q}_{p\to\infty}$. This means infinite linear combinations of terms $(p^b\zeta_p(\underline{\mathbf{s}}))$ with rational coefficients converge in $\mathbb{Q}_{p\to\infty}$ provided that the terms satisfy $b + |\underline{\mathbf{s}}| \to \infty$.

Definition 2.2. (Period map). The *completed finite period map* is the unique continuous ring homomorphism $\widehat{per} : \widehat{\mathcal{P}}^{\mathfrak{dr}} \to \mathbb{Q}_{p\to\infty}$ whose restriction to $\mathcal{P}^{\mathfrak{dr}}$ is $\alpha \mapsto (per_p(\alpha))$. Concretely, $\widehat{per}$ is given by

$$\widehat{per}\left(\sum_{i\geq 0} a_i \mathbb{L}^{n_i}\zeta^{\mathfrak{dr}}(\underline{\mathbf{s}}_i)\right) = \sum_{i\geq 0} a_i\big(p^{n_i}\zeta_p(\underline{\mathbf{s}}_i)\big) \in \mathbb{Q}_{p\to\infty}.$$

The map $\widehat{per}$ takes $\mathrm{Fil}^n\,\widehat{\mathcal{P}}^{\mathfrak{dr}}$ into $\mathrm{Fil}^n\mathbb{Q}_{p\to\infty}$. We expect that $\widehat{per}$ is compatible with the filtrations in a stronger sense, which would be an analog of the Grothendieck period conjecture. The conjecture takes the following form.

Conjecture 2.3. (Period conjecture). For every integer n,

$$\widehat{per}^{-1}\left(\mathrm{Fil}^n\mathbb{Q}_{p\to\infty}\right) = \mathrm{Fil}^n\,\widehat{\mathcal{P}}^{\mathfrak{dr}}.$$

In particular, $\widehat{per}$ is injective.

Conjecture 2.3 is equivalent to the statement that for every non-zero element

$$\sum_{i=1}^{n} a_i\mathbb{L}^{-|\underline{\mathbf{s}}_i|}\zeta^{\mathfrak{dr}}(\underline{\mathbf{s}}_i) \neq 0 \in \mathcal{P}^{\mathfrak{dr}},$$

the p-adic number

$$\sum_{i=1}^{n} a_i p^{-|\underline{\mathbf{s}}_i|}\zeta_p(\underline{\mathbf{s}}_i),$$

which is an a priori element of $\mathbb{Z}_p$ for all sufficiently large p, is actually in $\mathbb{Z}_p^\times$ for infinitely many p.

Definition 2.4. Let $\underline{s}$ be a composition. The *motivic multiple harmonic sum* is the following element of $\mathrm{Fil}^0\widehat{\mathcal{P}}^{\mathfrak{dr}}$:

$$H_{p-1}^{\mathfrak{dr}}(\underline{\mathbf{s}}) := \mathbb{L}^{-|\underline{\mathbf{s}}|}\sum_{i=0}^{k}\sum_{\ell_1,\dots,\ell_i\geq 0}\prod_{j=1}^{i}\binom{-s_j}{\ell_j}\zeta^{\mathfrak{dr}}(s_i+\ell_i,\dots,s_1+\ell_1)\zeta^{\mathfrak{dr}}(s_{i+1},\dots,s_k). \tag{2.6}$$

Formula (2.3) implies that

$$\widehat{per}(H_{p-1}^{\mathfrak{dr}}(\underline{\mathbf{s}})) = \big(H_{p-1}(\underline{\mathbf{s}})\big) \in \mathbb{Q}_{p\to\infty}.$$

3 The MHS algebra

A useful technique for proving supercongruences, used by the author in [24], is to express a quantity that appears in terms of multiple harmonic sums. For example, consider the hypergeometric sum

$$S_n := \sum_{k=0}^{n}\binom{n}{k}^4.$$

It is known that S_n satisfies a recurrence relation of length 3, which can be found with Zeilberger's algorithm. For $n = p$, we can compute

$$\begin{aligned} S_p &= 2+\sum_{k=1}^{p-1}\binom{p}{k}^4 \\ &= 2+\sum_{k=1}^{p-1}\frac{p^4}{k^4}\left[\left(1-\frac{p}{1}\right)\cdots\left(1-\frac{p}{k-1}\right)\right]^4 \\ &= 2+\sum_{k=1}^{p-1}\frac{p^4}{k^4}\left[\sum_{n\geq 0}(-1)^n p^n H_{k-1}(1^n)\right]^4. \end{aligned} \tag{3.1}$$

A product of multiple harmonic sums with the same limit of summation can be written as a linear combination of multiple harmonic sums with the same limit, for example, $H_n(1)H_n(2) = H_n(1,2) + H_n(2,1) + H_n(3)$. This is the so-called *series shuffle* or *stuffle* product. If we expand out the fourth power in (3.1) this way and concatenate a 4 at the beginning of each of the resulting compositions, we will find that there are integer coefficients $c_1, c_2, \ldots$ and compositions $\underline{\mathbf{s}}_1, \underline{\mathbf{s}}_2, \ldots$ with $|\underline{\mathbf{s}}_i| \to \infty$, such that

$$S_p = \sum_{i=1}^{\infty} c_i p^{|\underline{\mathbf{s}}_i|} H_{p-1}(\underline{\mathbf{s}}_i). \tag{3.2}$$

It is not hard to compute the first few terms to obtain

$$S_p \equiv 2 + p^4 H_{p-1}(4) - 4p^5 H_{p-1}(4,1) \mod p^6. \tag{3.3}$$

Now we can, for example, combine (3.3) with results of [30] to get a supercongruence for S_p in terms of Bernoulli numbers:

$$S_p \equiv 2 - \frac{16}{5} p^5 B_{p-5} \mod p^6.$$

We can combine (3.2) with (2.3) to obtain an expression for S_p in terms of p-adic multiple zeta values. The expression has many terms, but can be simplified using known relations among p-adic multiple zeta values to obtain a series that starts

$$S_p = 2 - 16\zeta_p(5) - 20\zeta_p(3)^2 - 143\zeta_p(7) - 456\zeta_p(3)\zeta_p(5) + \frac{696}{5}\zeta_p(5,3) + O(p^9).$$

Expansions like (3.2) are possible in a variety of other cases. The author makes the following definition in [24]:

Definition 3.1. The *MHS algebra* is the subset $\mathcal{M} \subset \mathbb{Q}_{p\to\infty}$ consisting of elements (a_p) such that there exist rational numbers $c_1, c_2, \ldots$, integers $b_1, b_2, \ldots$ going to infinity, and compositions $\underline{\mathbf{s}}_1, \underline{\mathbf{s}}_2, \ldots$, all independent of p, such that

$$(a_p) = \sum_{i=1}^{\infty} \left(c_i p^{b_i} H_{p-1}(\underline{\mathbf{s}}_i) \right) \in \mathbb{Q}_{p\to\infty}.$$

Concretely, this is equivalent to the condition that for every integer n, the congruence

$$a_p \equiv \sum_{\substack{i=1 \\ b_i < n}}^{\infty} c_i p^{b_i} H_{p-1}(\underline{\mathbf{s}}_i) \mod p^n \tag{3.4}$$

holds for all sufficiently large p (note that the sum of the right hand side of (3.4) is finite).

We sometimes abuse notation slightly and say that a quantity a_p (depending on p) is in the MHS algebra when we mean $(a_p) \in \mathbb{Q}_{p\to\infty}$ is in the MHS algebra. The computation (3.1) shows that the hypergeometric sum S_p is in the MHS algebra.

For each integer n, the quotient $\mathbb{Q}_{p\to\infty}/\mathrm{Fil}^n\mathbb{Q}_{p\to\infty}$ has the cardinality of the continuum. However, modulo $\mathrm{Fil}^n\mathbb{Q}_{p\to\infty}$, an element of the MHS algebra can be described by a finite amount of data, hence the image of the MHS algebra in $\mathbb{Q}_{p\to\infty}/\mathrm{Fil}^n\mathbb{Q}_{p\to\infty}$ is countable. It is perhaps surprising, then, to find that many familiar elementary quantities are in the MHS algebra. The following theorem records a few examples.

Theorem 3.2. ([24]) The following quantities are in the MHS algebra:

- the multiple harmonic sum $H_{f(p)}(\underline{\mathbf{s}})$, where $f(x) \in \mathbb{Z}[x]$ has positive leading coefficient and $\underline{\mathbf{s}}$ is a composition,
- the p-restricted multiple harmonic sum

$$H^{(p)}_{f(p)}(\underline{\mathbf{s}}) := \sum_{\substack{f(p)\geq n_1>\ldots>n_k\geq 1 \\ p\nmid n_1\cdots n_k}} \frac{1}{n_1^{s_1}\cdots n_k^{s_k}},$$

  where $f(x) \in \mathbb{Z}[x]$ has positive leading coefficient,
- the binomial coefficient $\binom{f(p)}{g(p)}$, for fixed polynomials $f(x), g(x) \in \mathbb{Z}[x]$ with positive leading coefficient,
- the Apéry numbers a_{p-1} and a_p, where

$$a_N := \sum_{n=0}^{N} \binom{N}{n}^2 \binom{N+n}{n}^2,$$

- for fixed r, k, the sum

$$\sum_{\substack{n_1+\ldots+n_k=p^r \\ p\nmid n_1\cdots n_k}} \frac{1}{n_1\ldots n_k}.$$

We prove the following result.

Theorem 3.3. The image of $\widehat{per}$ is exactly the MHS algebra.

Proof. Suppose (a_p) is in the MHS algebra, and let c_i, b_i, and $\underline{\mathbf{s}}_i$ be as in Definition 3.1. Then the infinite sum

$$a^{\mathfrak{dr}} := \sum_{i=1}^{\infty} c_i \mathbb{L}^{b_i} H^{\mathfrak{dr}}_{p-1}(\underline{\mathbf{s}}_i)$$

converges in $\widehat{\mathcal{P}}^{\mathfrak{dr}}$ because the i-th term is in $\mathrm{Fil}^{b_i}\widehat{\mathcal{P}}^{\mathfrak{dr}}$, and we have $\widehat{per}(a^{\mathfrak{dr}}) = (a_p)$.

Conversely, let

$$a^{\mathfrak{dr}} := \sum_{i=1}^{\infty} c_i \mathbb{L}^{b_i} \zeta^{\mathfrak{dr}}(\underline{\mathbf{s}}_i)$$

be an arbitrary element of $\widehat{\mathcal{P}}^{\mathfrak{dr}}$, with $c_i \in \mathbb{Q}$, $b_i \in \mathbb{Z}$, and $\underline{\mathbf{s}}_i$ compositions with $b_i + |\underline{\mathbf{s}}_i| \to \infty$. Yasuda [28] has shown that the sum of the terms in (2.3) with all $\ell_i = 0$, ranging over all compositions $\underline{\mathbf{s}}$, generate the space of multiple zeta values modulo $\zeta(2)$. Jarossay ([17], Proposition 3) has shown this implies that for each i, there exist rational coefficients $d_{i,1}, d_{i,2}, \ldots$ and compositions $\underline{\mathbf{t}}_{i,1}, \underline{\mathbf{t}}_{i,2}, \ldots$ with $|\underline{\mathbf{t}}_{i,j}| \geq |\underline{\mathbf{s}}_i|$ and $|\underline{\mathbf{t}}_{i,j}| \to \infty$, such that

$$\zeta^{\mathfrak{dr}}(\underline{\mathbf{s}}_i) = \sum_{j=1}^{\infty} d_{i,j} \mathbb{L}^{|\underline{\mathbf{t}}_{i,j}|} H^{\mathfrak{dr}}_{p-1}(\underline{\mathbf{t}}_{i,j}).$$

Then we have

$$\widehat{per}(a^{\mathfrak{dr}}) = \sum_{i,j} \left(c_i d_{i,j} p^{b_i + |\underline{\mathbf{t}}_{i,j}|} H_{p-1}(\underline{\mathbf{t}}_{i,j}) \right) \in \mathbb{Q}_{p\to\infty},$$

so that $\widehat{per}(a^{\mathfrak{dr}})$ is in the MHS algebra. ■

3.1 Motivic lifts

Definition 3.4. Suppose a_p is a quantity depending on p. A *motivic lift* of a_p is an element $a^{\mathfrak{dr}} \in \widehat{\mathcal{P}}^{\mathfrak{dr}}$ such that $\widehat{per}(a^{\mathfrak{dr}}) = (a_p)$.

Theorem 3.3 implies an element of $\mathbb{Q}_{p\to\infty}$ admits a motivic lift if and only if it is in the MHS algebra. Conjecture 2.3 would imply that motivic lifts are unique.

The motivic multiple harmonic sum $H^{\mathfrak{dr}}_{p-1}(\underline{\mathbf{s}})$ is a motivic lift of $H_{p-1}(\underline{\mathbf{s}})$. We can use $H^{\mathfrak{dr}}_{p-1}(\underline{\mathbf{s}})$ to write down motivic lifts of other elements of the MHS algebra. For example, if $k \geq r \geq 0$ are integers, we have an expression for the binomial coefficient

$$\binom{kp}{rp} = \binom{k}{r} \frac{\prod_{n=k-r}^{k-1} \left(\sum_{i \geq 0} n^i p^i H_{p-1}(1^i) \right)}{\prod_{n=0}^{r-1} \left(\sum_{i \geq 0} n^i p^i H_{p-1}(1^i) \right)}.$$

We define the *motivic binomial coefficient* to be

$$\binom{kp}{rp}^{\mathfrak{dr}} := \binom{k}{r} \frac{\prod_{n=k-r}^{k-1} \left(\sum_{i \geq 0} n^i \mathbb{L}^i H^{\mathfrak{dr}}_{p-1}(1^i) \right)}{\prod_{n=0}^{r-1} \left(\sum_{i \geq 0} n^i \mathbb{L}^i H^{\mathfrak{dr}}_{p-1}(1^i) \right)} \in \widehat{\mathcal{P}}^{\mathfrak{dr}}.$$

Note that each factor in the denominator is in $1 + \mathrm{Fil}^1 \widehat{\mathcal{P}}^{\mathfrak{dr}}$, so is invertible.

In some cases it is difficult to write down a motivic lift in closed form, but we can compute arbitrarily many terms with the aid of a computer. This is the case for a class of nested sums of binomial coefficients. We illustrate with an example. Suppose we encounter the quantity

$$\sum_{p-1 \geq n \geq m \geq 1} \binom{p+m}{m} \binom{p}{n}^2 \tag{3.5}$$

in a supercongruence. Through some manipulations, it can be shown that (3.5) is in the MHS algebra, though the expression is messy. With the help of a computer we find that the first few terms of a motivic lift of (3.5) are

$$3\zeta^{\mathfrak{dr}}(3) + 2\,\mathbb{L}\,\zeta^{\mathfrak{dr}}(3) - 2\mathbb{L}^2\zeta^{\mathfrak{dr}}(3) + \frac{53}{2}\zeta^{\mathfrak{dr}}(5) + 2\,\mathbb{L}^3\zeta^{\mathfrak{dr}}(3) + 17\,\mathbb{L}\,\zeta^{\mathfrak{dr}}(5) - \frac{1}{2}\zeta^{\mathfrak{dr}}(3)^2 \mod \mathrm{Fil}^7 \widehat{\mathcal{P}}^{\mathfrak{dr}}.$$

In other words we have

$$\begin{aligned} \sum_{p-1 \geq n \geq m \geq 1} \binom{p+m}{m} \binom{p}{n}^2 &\equiv 3\zeta_p(3) + 2p\zeta_p(3) - 2p^2\zeta_p(3) + \frac{53}{2}\zeta_p(5) \\ &+ 2p^3\zeta_p(3) + 17p\zeta_p(5) - \frac{1}{2}\zeta_p(3)^2 \mod p^7. \end{aligned}$$

To be explicit, the general recipe to write down a motivic lift of an element of the MHS algebra is as follows: given an element of the MHS algebra

$$(a_p) = \sum_{i=1}^{\infty} \left(c_i p^{b_i} H_{p-1}(\underline{\mathbf{s}}_i) \right) \in \mathbb{Q}_{p\to\infty},$$

a motivic lift of (a_p) is given by

$$a^{\mathfrak{dr}} := \sum_{i=1}^{\infty} c_i \mathbb{L}^{b_i} H_{p-1}^{\mathfrak{dr}}(\underline{\mathbf{s}}_i) \in \widehat{\mathcal{P}}^{\mathfrak{dr}}.$$

4 An algorithm for proving supercongruences

The author's work [24] gives an algorithm for finding and proving supercongruences between elements of the MHS algebra. Here, we state a version of this algorithm using motivic lifts.

Suppose we would like to prove a supercongruence

$$a_p \equiv a'_p \mod p^n, \tag{4.1}$$

where a_p and a'_p are in the MHS algebra.

1. We can find positive integers k, k', rational coefficients $c_1, \ldots, c_k$ and $c'_1, \ldots, c'_{k'}$, integers $b_1, \ldots, b_k$ and $b'_1, \ldots, b'_{k'}$, and compositions $\underline{\mathbf{s}}_1, \ldots, \underline{\mathbf{s}}_k$ and $\underline{\mathbf{s}}'_1, \ldots, \underline{\mathbf{s}}'_{k'}$ such that for all sufficiently large p,
$$a_p \equiv \sum_{i=1}^{k} c_i p^{b_i} H_{p-1}(\underline{\mathbf{s}}_i), \qquad a'_p \equiv \sum_{i=1}^{k'} c'_i p^{b'_i} H_{p-1}(\underline{\mathbf{s}}'_i) \qquad \mod p^n.$$
2. Use (2.6) to compute
$$\sum_{i=1}^{k} c_i \mathbb{L}^{b_i} H_{p-1}^{\mathfrak{dr}}(\underline{\mathbf{s}}_i), \quad \sum_{i=1}^{k'} c'_i \mathbb{L}^{b'_i} H_{p-1}^{\mathfrak{dr}}(\underline{\mathbf{s}}'_i) \in \widehat{\mathcal{P}}^{\mathfrak{dr}} \tag{4.2}$$
modulo $\mathrm{Fil}^n \widehat{\mathcal{P}}^{\mathfrak{dr}}$. This is a finite computation, and modulo $\mathrm{Fil}^n \widehat{\mathcal{P}}^{\mathfrak{dr}}$ there are unique representatives a and a' of (4.2) concentrated in weights less than n.
3. Use some source of relations between de Rham multiple zeta values to check whether $a = a'$. For example, we could use the double shuffle relations, which have been tabulated through weight 22 in the multiple zeta value data

mine [3] (our software uses data from the data mine). Another possibility would be to use a version of the techniques described in [5]. If $a = a'$, then we have found a proof of (4.1). If not, the truth of Conjecture 2.3 would imply that (4.1) fails for infinitely many p.

We provide software implementing this algorithm, which is described in Appendix A.

5 Galois theory of supercongreunces

As the coordinate ring of a pro-algebraic group G_{dR}, $\mathcal{P}^{\mathfrak{dr}}$ has the structure of a commutative Hopf algebra over $\mathbb{Q}$. The group G_{dR} acts on $\mathcal{P}^{\mathfrak{dr}}$ by algebra automorphisms in three different ways, coming from left multiplication, right multiplication, and conjugation. The completion $\widehat{\mathcal{P}}^{\mathfrak{dr}}$ has the structure of a complete Hopf algebra, and G_{dR} also acts on $\widehat{\mathcal{P}}^{\mathfrak{dr}}$ in three ways. The truth of the Period Conjecture 2.3 would imply that the MHS algebra $\mathcal{M} \subset \mathbb{Q}_{p\to\infty}$ also has a complete Hopf algebra structure and actions by G_{dR}.

Computing the actions of G_{dR} on $\mathcal{M}$ presents some computational challenges.

1. Given an element of the MHS algebra, it is sometimes difficult to write down an expansion in terms of MHS in closed form. It is even more difficult to write down a closed form motivic lift.
2. The result of the action is an element of $\mathcal{M}$ defined by an infinite linear combination of p-adic multiple zeta values, and it is not obvious whether this linear combination can be described combinatorially, for example, in terms of rational numbers defined by nested sums.

6 Computations in depth 1

Here, we explicitly compute a portion of the Galois action on elements of the MHS algebra in depth 1, that is, elements possessing a motivic lift in the closed subalgebra $\widehat{\mathcal{P}}_1^{\mathfrak{dr}}$ of $\widehat{\mathcal{P}}^{\mathfrak{dr}}$ generated by $\mathbb{L}$, $\mathbb{L}^{-1}$, and $\zeta^{\mathfrak{dr}}(n)$, $n \geq 3$.

There is a semi-direct factor $\mathbb{G}_m \subset G_{dR}$, and the action of $\mathbb{G}_m$ on $\widehat{\mathcal{P}}^{\mathfrak{dr}}$ by left multiplication comes from the weight grading. We denote this action by $\circ$, and an element $r \in \mathbb{G}_m(\mathbb{Q})$ acts by $r \circ \mathbb{L} = r\mathbb{L}$ and $r \circ \zeta^{\mathfrak{dr}}(\underline{\mathbf{s}}) = r^{|\underline{\mathbf{s}}|}\zeta^{\mathfrak{dr}}(\underline{\mathbf{s}})$.

We also make use of derivations, which come from the unipotent radical of G_{dR}. For each odd $k \geq 3$, we define δ_k to be the unique $\mathbb{Q}((\mathbb{L}))$-linear derivation on $\widehat{\mathcal{P}}_1^{\mathfrak{dr}}$

satisfying

$$\delta_k(\zeta^{\mathfrak{dr}}(m)) = \begin{cases} \mathbb{L}^m \text{ if } k = m, \\ 0 \text{ otherwise.} \end{cases}$$

These derivations are rescaled versions of the derivations appearing in [5] (§4), and they preserve weight.

Finally, there is a continuous ring homomorphism

$$\pi_{\mathbb{L}} : \widehat{\mathcal{P}}^{\mathfrak{dr}} \to \mathbb{Q}((\mathbb{L})),$$

sending $\mathbb{L}$ to $\mathbb{L}$ and sending $\zeta^{\mathfrak{dr}}(\underline{\mathbf{s}})$ to 0 for each non-empty composition $\underline{\mathbf{s}}$.

6.1 Depth 1 harmonic sums

First we consider the depth 1 multiple harmonic sums $H_{p-1}(n)$.

Proposition 6.1. For positive integers n and r, we have

$$\widehat{per}(r \circ H^{\mathfrak{dr}}_{p-1}(n)) = H^{(p)}_{rp}(n) := \sum_{\substack{j=1 \\ p \nmid j}}^{rp} \frac{1}{j^n}.$$

In addition we have $\pi_{\mathbb{L}}(H^{\mathfrak{dr}}_{p-1}(n)) = 0$.

Note that $H^{(p)}_{p}(n) = H_{p-1}(n)$, so $1 \in \mathbb{G}_m(\mathbb{Q})$ indeed acts trivially.

Proof. The motivic power sum is

$$H^{\mathfrak{dr}}_{p-1}(n) = (-1)^n \mathbb{L}^{-n} \sum_{k \geq 1} \binom{n+k-1}{n-1} \zeta^{\mathfrak{dr}}(n+k) \in \widehat{\mathcal{P}}^{\mathfrak{dr}}_1. \tag{6.1}$$

We act by r and apply $\widehat{per}$ to obtain

$$\widehat{per}(r \circ H^{\mathfrak{dr}}_{p-1}(n)) = (-1)^n p^{-n} \sum_{k \geq 1} \binom{n+k-1}{n-1} r^n \zeta_p(n+k).$$

It follows from [27] (Theorem 1) that the right hand side is $H^{(p)}_{rp}(n)$.

The second part of the claim is immediate from (6.1). ■

Next we compute the derivations.

Proposition 6.2. For positive integers n, k, with $k \geq 3$ odd, we have

$$\delta_k H^{\mathfrak{dr}}_{p-1}(n) = \begin{cases} (-1)^n \mathbb{L}^{k-n}\binom{k-1}{n-1} & : k > n, \\ 0 & : k \leq n. \end{cases}$$

Proof. Immediate from (6.1). ∎

6.2 Elementary symmetric sums

Here we consider the elementary symmetric multiple harmonic sums

$$H_{p-1}(1^n) := H_{p-1}(\underbrace{1, \ldots, 1}_{n}).$$

Remark 6.3. The association $H_{p-1}(\underline{\mathbf{s}}) \mapsto H^{\mathfrak{dr}}_{p-1}(\underline{\mathbf{s}})$ is a homomorphism for the series shuffle product ([16], Theorem 1). For example, we have $H_{p-1}(1)H_{p-1}(2) = H_{p-1}(1, 2) + H_{p-1}(2, 1) + H_{p-1}(3)$, and the corresponding identity $H^{\mathfrak{dr}}_{p-1}(1)H^{\mathfrak{dr}}_{p-1}(2) = H^{\mathfrak{dr}}_{p-1}(1,2) + H^{\mathfrak{dr}}_{p-1}(2,1) + H^{\mathfrak{dr}}_{p-1}(3)$ holds for the motivic lifts. Newton's formula relates the elementary symmetric and power sum symmetric functions, and Newton's formula also holds for the elementary symmetric and power sum motivic multiple harmonic sums.

Proposition 6.4. For positive integers n, r, we have

$$\widehat{per}(r \circ H^{\mathfrak{dr}}_{p-1}(1^n)) = H^{(p)}_{rp}(1^n) := \sum_{\substack{rp \geq m_1 > \ldots > m_n \geq 1 \\ p \nmid m_1 \ldots m_n}} \frac{1}{m_1 \ldots m_n}.$$

In addition we have $\pi_{\mathbb{L}}(H^{\mathfrak{dr}}_{p-1}(1^n)) = 0$.

Proof. Newton's formula for symmetric functions shows that we can write $H_{p-1}(1^n)$ as a polynomial in the depth 1 sums, independent of p. The result now follows from Proposition 6.1 and Remark 6.3. The second claim follows similarly. ∎

We similarly get a "finite" formula for the derivations.

Proposition 6.5. For positive integers n, k, with $k \geq 3$ odd, we have

$$\delta_k H^{\mathfrak{dr}}_{p-1}(1^n) = -\sum_{i=\max(1,k-n)}^{k-1} \frac{1}{k}\binom{k}{i}\mathbb{L}^i H^{\mathfrak{dr}}_{p-1}(1^{n-k+i}).$$

Proof. The proof is induction on n. For $n = 0$ this is true because both sides are 0.

Suppose $n \geq 1$. Newton's formula for symmetric functions implies

$$H^{\mathfrak{dr}}_{p-1}(1^n) = \frac{1}{n}\sum_{i=1}^{n}(-1)^{i-1}H^{\mathfrak{dr}}_{p-1}(i)H^{\mathfrak{dr}}_{p-1}(1^{n-i}).$$

We compute

$$\begin{aligned}\delta_k H^{\mathfrak{dr}}_{p-1}(1^n) &= \frac{1}{n}\sum_{i=1}^{n}(-1)^{i-1}\delta_k\big[H^{\mathfrak{dr}}_{p-1}(i)H^{\mathfrak{dr}}_{p-1}(1^{n-i})\big]\\ &= \frac{1}{n}\sum_{i=1}^{n}(-1)^{i-1}\delta_k[H^{\mathfrak{dr}}_{p-1}(i)]\cdot H^{\mathfrak{dr}}_{p-1}(1^{n-i}) + \frac{1}{n}\sum_{i=1}^{n}(-1)^{i-1}H^{\mathfrak{dr}}_{p-1}(i)\delta_k\big[H^{\mathfrak{dr}}_{p-1}(1^{n-i})\big]\end{aligned}$$

The left-hand sum is

$$-\frac{1}{n}\sum_{i=1}^{\min(n,k-1)}\binom{k-1}{i-1}\mathbb{L}^{k-i}H^{\mathfrak{dr}}_{p-1}(1^{n-i}) = -\frac{1}{n}\sum_{j=\max(1,k-n)}^{k-1}\binom{k-1}{j}\mathbb{L}^{j}H^{\mathfrak{dr}}_{p-1}(1^{n-k+j})$$

The right-hand sum is

$$\begin{aligned}&-\frac{1}{nk}\sum_{i=1}^{n}(-1)^{i-1}H^{\mathfrak{dr}}_{p-1}(i)\sum_{j=\max(1,k-n+i)}^{k-1}\binom{k}{j}\mathbb{L}^j H^{\mathfrak{dr}}_{p-1}(1^{n-i-k+j})\\ &= -\frac{1}{nk}\sum_{j=\max(1,k-n+1)}^{k-1}\binom{k}{j}\mathbb{L}^j\sum_{i=1}^{n-k-j}(-1)^{i-1}H^{\mathfrak{dr}}_{p-1}(i)H^{\mathfrak{dr}}_{p-1}(1^{n-k+j-i})\\ &= -\frac{1}{nk}\sum_{j=\max(1,k-n+1)}^{k-1}\binom{k}{j}\mathbb{L}^j(n-k+j)H^{\mathfrak{dr}}_{p-1}(1^{n-k+j}).\end{aligned}$$

The result now follows by adding these two sums together and observing that the coefficient of $\mathbb{L}^j H^{\mathfrak{dr}}_{p-1}(1^{n-k+j})$ is

$$-\frac{1}{n}\binom{k-1}{j} - \frac{1}{nk}\binom{k}{j}(n-k+j) = \frac{-1}{k}\binom{k}{j}.$$

■

6.3 Binomial coefficients

Next we compute the action on the binomial coefficients $\binom{ap}{bp}$, which are in MHS algebra.

First, define

$$c_n := \binom{np}{p} = n\sum_{j\geq 0}(n-1)^j p^j H_{p-1}(1^j),$$

with motivic lift

$$c_n^{\mathfrak{dr}} := n\sum_{j\geq 0}(n-1)^j \mathbb{L}^j H^{\mathfrak{dr}}_{p-1}(1^j) \in \widehat{\mathcal{P}}_1^{\mathfrak{dr}}. \tag{6.2}$$

Proposition 6.6. Fix a positive integer $r \in \mathbb{G}_m(\mathbb{Q})$ and a positive integer n. Then

$$\widehat{per}(r \circ c_n^{\mathfrak{dr}}) = \frac{n}{\binom{rn}{r}}\binom{rnp}{rp}.$$

In addition, we have $\pi_{\mathbb{L}}(c_n^{\mathfrak{dr}}) = n$.

Proof. We compute

$$\begin{aligned}\widehat{per}(r \circ c_n^{\mathfrak{dr}}) &= n\sum_{j\geq 0}(n-1)^j r^j p^j H^{(p)}_{rp}(1^j)\\ &= n\prod_{\substack{i=1\\ p\nmid i}}^{rp}\left((1+\frac{(n-1)rp}{i}\right)\\ &= \frac{n}{\binom{rn}{r}}\binom{rnp}{rp}.\end{aligned}$$

The second part of the claim follows from the expression for c_n in terms of the elementary symmetric sums $H_{p-1}(1^k)$. ■

Proposition 6.7. For a positive integer $r \in \mathbb{G}_m(\mathbb{Q})$ and $a \geq b \geq 0$, we have

$$\widehat{per}\left(r \circ \binom{ap}{bp}^{\mathfrak{dr}}\right) = \frac{\binom{a}{b}}{\binom{ra}{rb}} \binom{rap}{rbp}.$$

In addition, we have $\pi_{\mathbb{L}}(\binom{ap}{bp}^{\mathfrak{dr}}) = \binom{a}{b}$.

Proof. First we observe that

$$\binom{ap}{bp}^{\mathfrak{dr}} = \frac{c_a^{\mathfrak{dr}} \cdots c_{a-b+1}^{\mathfrak{dr}}}{c_b^{\mathfrak{dr}} \cdots c_1^{\mathfrak{dr}}}. \tag{6.3}$$

It follows that

$$\widehat{per}\left(r \circ \binom{ap}{bp}^{\mathfrak{dr}}\right) = \frac{\prod_{n=a-b+1}^{a} n\binom{rnp}{rp}/\binom{rn}{r}}{\prod_{n=1}^{b} n\binom{rnp}{rp}/\binom{rn}{r}}$$
$$= \frac{\binom{a}{b}}{\binom{ra}{rb}} \binom{rap}{rbp}.$$

The second part of the claim follows from (6.3) and Proposition 6.6. ■

We can also compute how the derivations act on binomial coefficients.

Proposition 6.8. For positive integers a, b, and k, with $k \geq 3$ odd, we have

$$\delta_k \binom{ap}{bp}^{\mathfrak{dr}} = -\frac{\mathbb{L}^k}{k}\left[a^k - b^k - (a-b)^k\right]\binom{ap}{bp}^{\mathfrak{dr}}.$$

Proof. Proposition 6.5 and (6.2) imply

$$\delta_k c_n^{\mathfrak{dr}} = -\frac{n}{k}\sum_{j\geq 0}(n-1)^j \mathbb{L}^j \sum_{i=\max(1,k-j)}^{k-1}\binom{k}{i} p^i H_{p-1}^{\mathfrak{dr}}(1^{j-k+i})$$
$$= -\frac{n}{k}\sum_{m\geq 0} H_{p-1}^{\mathfrak{dr}}(1^m)\sum_{j=m+1}^{m+k-1}\binom{k}{j-m}(n-1)^j \mathbb{L}^{k+m}$$
$$= -\frac{np^k}{k}\sum_{m\geq 0}(n-1)^m H_{p-1}^{\mathfrak{dr}}(1^m)\sum_{j=1}^{k-1}\binom{k}{j}(n-1)^j$$
$$= -\frac{\mathbb{L}^k}{k}\left[n^k - (n-1)^k - 1\right]c_n.$$

Then we have

$$\delta_k \binom{ap}{bp}^{\mathfrak{dr}} = \binom{ap}{bp}^{\mathfrak{dr}} \cdot \left(\sum_{n=a-b+1}^{a} \frac{\delta_k c_n^{\mathfrak{dr}}}{c_n^{\mathfrak{dr}}} - \sum_{n=1}^{b} \frac{\delta_k c_n^{\mathfrak{dr}}}{c_n^{\mathfrak{dr}}} \right)$$
$$= -\frac{\mathbb{L}^k}{k}\left[a^k - b^k - (a-b)^k\right] \binom{ap}{bp}^{\mathfrak{dr}}.$$

■

6.4 Products of factorials

For b a positive integer, we do not expect that $(bp)!$ is in the MHS algebra. However, if $b_1, \ldots, b_k$ are positive integers and $n_1, \ldots, n_k$ are arbitrary integers satisfying $\sum n_i b_i = 0$, then the product

$$\alpha := \prod_{i=1}^{k} (b_i p)!^{n_i}$$

is in the MHS algebra because we have

$$\alpha = \frac{\alpha}{(p!)^{n_1 b_1 + \ldots + n_k b_k}} = \prod_{i=1}^{k} \left(\frac{(b_i p)!}{p!^{b_i}} \right)^{n_i} = \prod_{i=1}^{k} (c_1 c_2 \ldots c_{b_i})^{n_i}. \tag{6.4}$$

We get a motivic lift $\alpha^{\mathfrak{dr}}$ by replacing each term c_j with $c_j^{\mathfrak{dr}}$.

Proposition 6.9. For r a positive integer, we have

$$\widehat{per}(r \circ \alpha^{\mathfrak{dr}}) = \prod_{i=1}^{k} \left(\frac{b_i!\, r!^{b_i}}{(rb_i)!} (rb_i p)! \right)^{n_i}.$$

Additionally we have $\pi_{\mathbb{L}}(\alpha^{\mathfrak{dr}}) = \prod_i b_i!^{n_i}$.

Proof. This is straightforward to derive from (6.4) and Proposition 6.6. ■

Proposition 6.10. For $m \geq 3$ odd, we have

$$\frac{\delta_m \alpha^{\mathfrak{dr}}}{\alpha^{\mathfrak{dr}}} = -\frac{\mathbb{L}^m}{m} \sum_{i=1}^{k} n_i b_i^m.$$

Proof. We compute

$$\frac{\delta_m \alpha^{\mathfrak{dr}}}{\alpha^{\mathfrak{dr}}} = \sum_{i=1}^{k} n_i \left(\frac{\delta_m c_1^{\mathfrak{dr}}}{c_1^{\mathfrak{dr}}} + \cdots + \frac{\delta_m c_{b_i}^{\mathfrak{dr}}}{c_{b_i}^{\mathfrak{dr}}} \right)$$
$$= -\frac{\mathbb{L}^m}{m} \sum_{i=1}^{k} n_i \left(b_i^m - b_i \right)$$
$$= -\frac{\mathbb{L}^m}{m} \sum_{i=1}^{k} n_i b_i^m,$$

where on the second line we observed that each summand telescopes, and on the last line we used that $\sum_i n_i b_i = 0$. ■

6.5 A supercongruence for factorials

We end with an alternate proof of a known supercongruence for factorials. Our proof uses the derivations δ_k.

Theorem 6.11. (Granville [13], Proposition 5). Suppose $b_1, \ldots, b_k \in \mathbb{Z}_{\geq 0}$, $n_1, \ldots, n_k \in \mathbb{Z}$, and assume that

$$\sum_{i=1}^{k} n_i b_i^m = 0$$

for $m = 1, 3, 5, \ldots, 2n-1$. Then for all primes p sufficiently large,

$$\prod_{i=1}^{k} (b_i p)!^{n_i} \equiv \prod_{i=1}^{k} b_i!^{n_i} \mod p^{2n+1}. \tag{6.5}$$

Proof. Let $\alpha^{\mathfrak{dr}}$ be the motivic lift of the left hand side of (6.5) coming from (6.4). By Proposition 6.10, we have $\delta_m \alpha^{\mathfrak{dr}} = 0$ for $m = 3, 5, \ldots, 2n-1$. Proposition 6.8 implies $\delta_m \alpha^{\mathfrak{dr}} \in \mathrm{Fil}^m \widehat{\mathcal{P}}^{\mathfrak{dr}}$ for all m, so we conclude $\delta_m \alpha^{\mathfrak{dr}} \in \mathrm{Fil}^{2n+1} \widehat{\mathcal{P}}^{\mathfrak{dr}}$ for all m. It follows that

$$\alpha^{\mathfrak{dr}} \equiv \pi_{\mathbb{L}}(\alpha^{\mathfrak{dr}}) = \prod_{i=1}^{k} b_i!^{n_i} \mod \mathrm{Fil}^{2n+1} \widehat{\mathcal{P}}^{\mathfrak{dr}}.$$

We obtain the desired result by applying $\widehat{per}$. ■

Appendix A. Software

We provide software for computing motivic lifts and for verifying supercongruences. The software is written in Python 2.7, and is available at https://sites.google.com/site/julianrosen/mhs.

The software can express multiple harmonic sums interms of p-adic multiple zeta values using a chosen basis.

```
>>> a = Hp(1,3,2)
>>> a.disp()
```

$H_{p-1}(1,3,2)$

```
>>> a.mzv()
```

$$-\frac{9}{2}p^{-6}\zeta_p(3)^2+\frac{67}{16}p^{-6}\zeta_p(7)+\frac{23}{2}p^{-6}\zeta_p(3)\zeta_p(5)+p^{-6}\zeta_p(5,3)+O(p^3)$$

The software will also compute expansions for other elements of the MHS algebra, for example, the Apéry numbers.

```
>>> a = aperybp()
>>> a.disp()
```

$\sum_{n=0}^{p-1}\binom{p-1}{n}^2\binom{p+n-1}{n}^2$

```
>>> a.mzv(err=8)
```

$$1+2\zeta_p(3)-16\zeta_p(5)+4\zeta_p(3)^2-100\zeta_p(7)+O(p^8)$$

The optional argument err=8 says we only want an expansion modulo p^8. We can also compute expansions for p-restricted multiple harmonic sums whose limit of summation is polynomial in p.

```
>>> a = H_poly_pr([1,-2,2], (1,2))
>>> a.disp()
```

$H^{(p)}_{2p^2-2p+1}(1,2)$

```
>>> a.mzv(err=4)
```

$$6p^{-3}\zeta_p(3)-10p^{-2}\zeta_p(3)-4p^{-1}\zeta_p(3)+27p^{-3}\zeta_p(5)-\frac{293}{3}p^{-2}\zeta_p(5)+8p^{-3}\zeta_p(3)^2+O(p^4)$$

We can also deal with various other nested sums. For example, suppose we are interested in the sum

$$\sum_{n=1}^{p-3} \binom{p}{n}^2 \frac{1}{(n+1)(n+2)}.$$

```
>>> a = (BINN(0,0)**2*nn(-1)*nn(-2)).sum(1,-2).e_p()
>>> a.mzv()
```

$$-\frac{11}{8}p^2 + \frac{35}{16}p^3 - 2\zeta_p(3) - \frac{93}{32}p^4 + \frac{215}{64}p^5 + \frac{5}{2}p^2\zeta_p(3) - 6\zeta_p(5) + O(p^6)$$

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References

[1] André, Y. Galois theory, motives and transcendental numbers. In *Renormalization and Galois Theories,* volume 15 of *IRMA Lect. Math. Theor. Phys.*, 165–77. Zürich: Eur. Math. Soc., 2009.

[2] Apéry, R. "Irrationalité de ζ (2) et ζ (3)."*Astérisque* 61, no. 11–13 (2012): 1.

[3] Blümlein, J., D. J. Broadhurst, and J. A. M. Vermaseren. "The multiple zeta value data mine." *Computer Physics Communications* 181, no. 3 (2010): 582–625.

[4] Brown, F. "Mixed Tate motives over $\mathbb{Z}$." *Ann. of Math. (2)* 175, no. 2 (2012): 949–76.

[5] Brown, F. C. S. On the decomposition of motivic multiple zeta values. In *Galois–Teichmüller theory and arithmetic geometry,* volume 63 of *Adv. Stud. Pure Math.*, 31–58. Japan, Tokyo: Math. Soc., 2012.

[6] Brown, F. Single-valued motivic periods and multiple zeta values. In *Forum of Mathematics, Sigma,* volume 2. Cambridge University Press, 2014.

[7] Chatzistamatiou, A. "On integrality of p-adic iterated integrals." *Journal of Algebra* 474 (2017): 240–70.

[8] Chowla, S., J. Cowles, and M. Cowles. "Congruence properties of Apéry numbers." *Journal of Number Theory* 12, no. 2 (1980): 188–90.

[9] Deligne, P. Periods for the fundamental group. *Arizona winter school 2002*, *course notes and project description*, 2002.

[10] Deligne, P. and A. B. Goncharov. "Groupes fondamentaux motiviques de Tate mixte." *Ann. Sci. École Norm. Sup. (4)* 38, no. 1 (2005): 1–56.

[11] Furusho, H. "p-adic multiple zeta values. I. p-adic multiple polylogarithms and the p-adic KZ equation." *Invent. Math.* 155, no. 2 (2004): 253–86.

[12] Furusho, H. "p-adic multiple zeta values. II. Tannakian interpretations." *Amer. J. Math.* 129, no. 4 (2007): 1105–44.

[13] Granville, A. A decomposition of Riemann's zeta-function. In *Analytic number theory (Kyoto, 1996)*, volume 247 of *London Math. Soc. Lecture Note Ser.*, 95–101. Cambridge: Cambridge Univ. Press, 1997.

[14] Huber, A. and S. Müller–Stach. *Periods and Nori Motives,* volume 65 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Berlin Heidelberg: Springer, 2017.

[15] Imatomi, K.,M. Kaneko, and E. Takeda. "Multi-poly-Bernoulli numbers and finite multiple zeta values." *J. Integer Seq.* 17, no. 4 (2014): Article 14.4.5, 12.

[16] Jarossay, D. "Algebraic relations, Taylor coefficients of hyperlogarithms and images by Frobenius - I: The prime multiple harmonic sum motive." (2016): preprint arXiv:1412.5099.

[17] Jarossay, D. "Algebraic relations, Taylor coefficients of hyperlogarithms and images by Frobenius - II: Relations with other motives and the Taylor period map." (2016): preprint arXiv:1601.01158.

[18] Jarossay, D. "An explicit theory of $\pi_1^{un,crys}(\mathbb{P}^1 - \{0, 1, \infty\})$: summary of parts I and II." (2016): preprint arXiv:1611.01921.

[19] Murahara, H. and M. Sakata. "On multiple zeta values and finite multiple zeta values of maximal height." 2016, arXiv:1612.04071.

[20] Murahara H. "Derivation relations for finite multiple zeta values." *International Journal of Number Theory* (2015): 1–9.

[21] Ono, M. "Finite multiple zeta values associated with 2-colored rooted trees." (2016): preprint arXiv:1609.09168.

[22] Oyama, K. "Ohno's relation for finite multiple zeta values." (2015): preprint arXiv:1506.00833.

[23] Rosen, J. "Asymptotic relations for truncated multiple zeta values." *J. London Math. Soc.* 91, no. 2 (2015): 554–72.

[24] Rosen, J. "The MHS algebra and supercongruences." (2016): preprint arXiv:1608.06864.

[25] Saito, S. and N. Wakabayashi. "Sum formula for finite multiple zeta values." *Journal of the Mathematical Society of Japan* 67, no. 3 (2015): 1069–76.

[26] Saito, S. and N. Wakabayashi. "Bowman–Bradley type theorem for finite multiple zeta values." *Tohoku Mathematical Journal* 68, no. 2 (2016): 241–51.

[27] Washington, L. C. "p-adic L-functions and sums of powers." *J. Number Theory* 69, no. 1 (1998): 50–61.

[28] Yasuda, S. "Finite real multiple zeta values generate the wholse space z." (2016): preprint arXiv:1403.1134.

[29] Yasuda, S. *Two conjectures on p-adic mzv and truncated multiple harmonic sums (slides)*. 2014.

[30] Zhao, J. "Wolstenholme type theorem for multiple harmonic sums." *Int. J. Number Theory* 4, no. 1 (2008): 73–106.

[31] Zhao, J. "Finite multiple zeta values and finite euler sums." (2015): preprint arXiv:1507.04917.