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Asymptotic relations for truncated multiple zeta values

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Abstract

Multiple zeta values (MZVs) are real numbers defined by an infinite series, generalizing values of the Riemann zeta function at positive integers. Finite truncations of this series are called multiple harmonic sums and are known to have interesting arithmetic properties. When the truncation point is one less than a prime p, the modulo p values of multiple harmonic sums are called finite MZVs. This paper introduces a new class of congruence for multiple harmonic sums, which we call *weighted congruences*. These congruences can hold modulo arbitrarily large powers of p, and generalize the class of homogeneous relations for finite MZVs. Unlike results for finite MZVs, weighted congruences typically involve harmonic sums of multiple weights, which are multiplied by explicit powers of p depending on weight. We also introduce a class of p-adic series identities, which we call *asymptotic relations*, giving weighted congruences holding modulo arbitrarily large powers of p. We define a new truncated analogue of the multiple zeta function, and use it to give an algebraic framework for studying weighted congruences and asymptotic relations.

1. *Introduction*

The Riemann zeta function is defined by the infinite series

$$\zeta(s) := \sum_{n\geqslant 1} \frac{1}{n^s},$$

which converges provided that $\mathrm{Re}(s) > 1$. Values of this function are particularly interesting when $s \geqslant 2$ is an integer. In 1735, Euler computed

$$\zeta(2) = \frac{\pi^2}{6}.$$

Euler was actually able to obtain all of the even zeta values in terms of Bernoulli numbers, which are rational numbers defined by a generating function:

$$\zeta(2k) = \frac{(-1)^{k-1}2^{2k-1}B_{2k}}{(2k)!}\pi^{2k}, \quad \sum_{n\geqslant 0} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1}.$$

In particular, $\zeta(2k)$ is a rational multiple of π^{2k}, so lies in $\mathbb{Q}[\zeta(2)]$. Lindemann proved in 1882 that π is transcendental, so the algebraic structure of $\zeta(2), \zeta(4), \ldots$ is completely determined.

Much less is known about the odd zeta values. Apéry [**1**] proved that $\zeta(3)$ is irrational, and Rivoal [**11**] showed that infinitely many of the odd zeta values are irrational. Zudilin [**19**] proved that at least one of the four numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, or $\zeta(11)$ is irrational. It is conjectured that $\zeta(3), \zeta(5), \ldots$ are algebraically independent over $\mathbb{Q}(\pi)$, but at present this conjecture is inaccessible.

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1.1. *Multiple zeta values*

For $k \geqslant 1$, the *multiple zeta function of depth* k is defined by

$$\zeta(s_1, \dots, s_k) := \sum_{n_1 > \cdots > n_k \geqslant 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}.$$

For $k = 1$, this is just the Riemann zeta function. Values of the multiple zeta function for $s_1, \dots, s_k$ integers are real numbers called *multiple zeta values* (*MZVs*) or *Euler–Zagier sums*. To ensure convergence, we insist $s_1 \geqslant 2$ and $s_2, \dots, s_k \geqslant 1$.

MZVs satisfy many algebraic relations. For example, $\zeta(2, 1) = \zeta(3)$, a fact that was known to Euler. Another family of examples is given by the sum formula, which says that for all positive integers $k < n$,

$$\sum_{\substack{s_1 + \cdots + s_k = n \\ s_1 \geqslant 2, s_2, \dots, s_k \geqslant 1}} \zeta(s_1, \dots, s_k) = \zeta(n).$$

This relation was conjectured by Moen (see [5]) and proved independently by Zagier and by Granville [3]. The following problem is well known.

Problem A. Determine the set of relations satisfied by the MZVs.

It can be shown by expanding out a product of nested sums that the product of two MZVs is a linear combination (with integer coefficients) of MZVs. This is called a quasi-shuffle relation, and it shows that to solve Problem A it suffices to consider *linear* relations. Also, both of the relations given above are homogeneous (that is, the *weight* $s_1 + \cdots + s_k$ is constant among all terms $\zeta(s_1, \dots, s_k)$ appearing in each relation). It is conjectured that the space spanned by MZVs is graded by weight, meaning that every relation among MZVs can be decomposed into homogeneous relations. A general method for generating relations among MZVs is given by the extended double shuffle relations [8], described briefly in Subsection 2.2. It is conjectured that every relation is a consequence of the extended double shuffle relations [8, Conjecture 1].

1.2. *Multiple harmonic sums*

Truncations of the MZV series are called multiple harmonic sums, and they have interesting arithmetic properties. Recall that a *composition* is a finite ordered list $\mathbf{s} = (s_1, \dots, s_k)$ of positive integers. We define the *weight* and *depth* (or *length*) of a composition $\mathbf{s}$ by $w(\mathbf{s}) = s_1 + \cdots + s_k$ and $\ell(\mathbf{s}) = k$, respectively. We allow the empty composition $\emptyset$.

Definition 1.1. Let $\mathbf{s} = (s_1, \dots, s_k)$ be a composition and n be a positive integer. We define the *multiple harmonic sum*

$$H_n(\mathbf{s}) = H_n(s_1, \dots, s_k) := \sum_{n \geqslant n_1 > \cdots > n_k \geqslant 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}} \in \mathbb{Q}.$$

By convention, we take $H_n(\emptyset) = 1$.

Multiple harmonic sums are known to satisfy many congruences when $n = p - 1$, p a prime. For example, Wolstenholme's Theorem is the old result that the congruence

$$H_{p-1}(1) \equiv 0 \mod p^2 \tag{1.1}$$

holds for $p \geqslant 5$ (for r a rational number and p a prime, we write $r \equiv 0 \mod p^n$ to mean that p^n divides the numerator of r; that is, $v_p(r) \geqslant n$, where v_p is the p-adic valuation). As in this example, it is typical for congruences involving multiple harmonic sums to fail for a finite number of small primes.

Systematic study of the mod p structure of multiple harmonic sums was undertaken in independent works of Hoffman [7] and Zhao [18]. A recent formulation of the problem (see the forthcoming paper 'Finite multiple zeta values' by Kaneko and Zagier) uses the ring

$$\mathcal{A} := \frac{\prod_p \mathbb{Z}/p\mathbb{Z}}{\bigoplus_p \mathbb{Z}/p\mathbb{Z}},$$

which is an algebra over $\mathbb{Q}$. An element of $\mathcal{A}$ consists of a family of residue classes $a_p \in \mathbb{Z}/p\mathbb{Z}$ for all p, where two families (a_p) and (b_p) are identified if $\{p : a_p \not\equiv b_p\}$ is a finite set.

Definition 1.2. For integers $s_1, \ldots, s_k \geqslant 1$, the *finite MZV* is given by

$$\zeta_{\mathcal{A}}(s_1, \ldots, s_k) := (H_{p-1}(s_1, \ldots, s_k) \mod p) \in \mathcal{A}.$$

Congruences modulo p for multiple harmonic sums $H_{p-1}(\mathbf{s})$ holding for all sufficiently large p correspond to relations satisfied by the finite MZVs. Wolstenholme's congruence (1.1), for instance, implies $\zeta_{\mathcal{A}}(1) = 0$. The analogue of Problem A is the following.

Problem B. Determine the set of relations satisfied by the finite MZVs.

As with the ordinary MZVs, it suffices to consider linear relations. It is also believed that the space of finite MZVs is graded by weight.

1.3. *Weighted congruences and asymptotic relations*

This paper introduces two new, related classes of identities for multiple harmonic sums.

1.3.1. *Weighted congruences.* Wolstenholme's congruence (1.1) can be extended. For all $p \geqslant 5$, we have the congruence

$$H_{p-1}(1) + \frac{1}{3}p^2 H_{p-1}(2,1) \equiv 0 \mod p^4, \tag{1.2}$$

from which (1.1) follows by reducing modulo p^2 (that this congruence holds modulo p^3 can be shown using known expressions for $H_{p-1}(1)$ and $H_{p-1}(2,1)$ in terms of Bernoulli numbers; the specific form of the expressions can be found in [18]). We extend this congruence further, adding an additional multiple harmonic sum to obtain

$$H_{p-1}(1) + \tfrac{1}{3}p^2 H_{p-1}(2,1) - \tfrac{1}{6}p^4 H_{p-1}(4,1) \equiv 0 \mod p^5, \tag{1.3}$$

which holds for all $p \geqslant 5$. This is proved in Paragraph 5.2.1. This congruence differs from results for finite MZVs in that each multiple harmonic sum is multiplied by an explicit power of p depending on weight (we call this a *weighting*). The following definition is new.

Definition 1.3. A *weighted congruence for multiple harmonic sums* (or simply a *weighted congruence*) is a congruence of the form

$$\sum_{w(\mathbf{s})<n} \alpha_{\mathbf{s}} p^{w(\mathbf{s})} H_{p-1}(\mathbf{s}) \equiv 0 \mod p^n, \tag{1.4}$$

which holds for all sufficiently large p, where n is a positive integer; the sum is over compositions $\mathbf{s}$ of weight less than n, and the coefficients $\alpha_{\mathbf{s}} \in \mathbb{Q}$ do not depend on p.

Any congruence that can be put in the form (1.4) through multiplication by a power of p will also be called weighted. For instance, the congruence (1.3) is weighted because multiplication by p puts it in the form (1.4) for $n = 6$, with $\alpha_{(1)} = 1$, $\alpha_{(2,1)} = \frac{1}{3}$, $\alpha_{(4,1)} = -\frac{1}{6}$, and all other $\alpha_{\mathbf{s}}$ zero. We pose the following variation of Problem B.

Problem C. Determine the set of weighted congruences.

The weighted congruences include all *homogeneous* relations among the finite MZVs, that is, all congruences of the form

$$\sum_{w(\mathbf{s})=n} \alpha_{\mathbf{s}} H_{p-1}(\mathbf{s}) \equiv 0 \mod p \quad \text{for } p \text{ large.}$$

Therefore, if we assume the conjecture that the space of finite MZVs is graded by weight, then a solution to Problem C would give a solution to Problem B. Unlike what is conjectured for finite MZVs, weighted congruences can be essentially non-homogeneous (as the congruence (1.3) illustrates).

1.3.2. *Asymptotic relations.* One often finds that a weighted congruence can be extended to a stronger one, as we saw above with (1.1) extended to (1.2), and then to (1.3). In this example and in many others, extensions can be continued indefinitely. The following definition is new.

Definition 1.4. An *asymptotic relation for weighted multiple harmonic sums* (or simply an *asymptotic relation*) is a formal infinite sum

$$\sum_{\mathbf{s}} \alpha_{\mathbf{s}}\, p^{w(\mathbf{s})} H_{p-1}(\mathbf{s}), \tag{1.5}$$

with coefficients $\alpha_{\mathbf{s}} \in \mathbb{Q}$ not depending on p, such that for every positive integer n, the weighted congruence

$$\sum_{w(\mathbf{s})<n} \alpha_{\mathbf{s}}\, p^{w(\mathbf{s})} H_{p-1}(\mathbf{s}) \equiv 0 \mod p^n \tag{1.6}$$

holds for all p sufficiently large. We say that the asymptotic relation (1.5) is an *asymptotic extension* of each of the weighted congruences (1.6).

It turns out that asymptotic relations are fairly common. In Section 4, we derive two infinite families of asymptotic relations. We pose the following problem.

Problem D. Determine the set of asymptotic relations.

We conjecture that every weighted congruence admits an asymptotic extension (see Conjecture A), in which case solving Problem D would give a solution to Problem C.

1.4. *Results*

The present work introduces a new analogue of the multiple zeta function, and uses it to establish an algebraic framework for investigating weighted congruences and asymptotic relations. We use two topological rings: a completion of the ring of quasi-symmetric functions, and a projective limit of rings related to the ring $\mathcal{A}$ used to study finite MZVs. We discuss a known framework for MZVs in Section 2, and we describe our modified framework in Section 3.

In Section 4, we prove two results. The first result, stated as Theorem 4.1 and in another form as Theorem 4.3, gives an asymptotic extension of the known congruence

$$H_{p-1}(s_1, \dots, s_k) \equiv (-1)^{s_1+\cdots+s_k} H_{p-1}(s_k, \dots, s_1) \mod p,$$

which holds for all compositions $(s_1, \dots, s_k)$ and all p. The second result, Theorem 4.5, gives asymptotic extensions of a duality result for finite MZVs, proved by Hoffman [**7**] and extended by Zhao [**18**].

In Section 5, we use the results of Section 4 to derive congruences holding modulo high powers of p. We provide a script automating our method of computation for the computer algebra system Mathematica.

In Section 6, we consider various quantities appearing in arithmetic that can be expressed in terms of weighted multiple harmonic sums. These quantities include values of the p-adic zeta function at positive integers.

1.5. *Recent related work*

Several recent results establish homogeneous relations for finite MZVs. Hessami Pilehrood, Hessami Pilehrood, and Tauraso [**4**] showed (among other interesting results) that for all positive integers a, b, the congruence

$$H_{p-1}(\{2\}^a, 3, \{2\}^b) \equiv \frac{(-1)^{a+b}(a-b)}{(a+1)(b+1)} \binom{2a+2b+2}{2a+1} B_{p-2a-2b-3} \mod p$$

holds for $p > 2a + 2b + 3$. There are known expressions relating Bernoulli numbers to depth 2 finite MZVs (see [**18**]). Although it requires some choices, we can use these expressions to transform the congruence above into the relation

$$\zeta_{\mathcal{A}}(\{2\}^a, 3, \{2\}^b) + (-1)^{a+b}\frac{2(a-b)}{b+1}\zeta_{\mathcal{A}}(2a+2, 2b+1) = 0.$$

Linebarger and Zhao [**9**] recently generalized some of the results of [**4**].

Recent work of Saito and Wakabayashi concerns finite MZVs. In [**16**], a conjecture of Kaneko is resolved: for all positive integers $k < w$, the congruence

$$\sum_{\substack{s_1 \geqslant 2, s_2, \ldots, s_k \geqslant 1 \\ s_1 + \cdots + s_k = w}} H_{p-1}(s_1, \ldots, s_k) \equiv \left(1 + (-1)^k \binom{w-1}{k-1}\right) \frac{B_{p-w}}{w} \mod p$$

holds for p sufficiently large. Again using depth 2 finite MZVs, this can be put in the pleasant form

$$\sum_{\substack{s_1 \geqslant 2, s_2, \ldots, s_k \geqslant 1 \\ s_1 + \cdots + s_k = w}} \zeta_{\mathcal{A}}(s_1, \ldots, s_k) = \frac{k}{w}\zeta_{\mathcal{A}}(k, w-k) - \frac{1}{w}\zeta_{\mathcal{A}}(1, w-1).$$

See also [**15**] for more results on finite MZVs.

2. *MZVs and Hoffman's algebra*

A *quasi-symmetric* function over $\mathbb{Q}$ is a formal power series of bounded degree in the variables $x_1, x_2, \ldots$, having coefficients in $\mathbb{Q}$, such that the coefficient of $x_{i_1}^{s_1} \cdots x_{i_k}^{s_k}$ is the same as the coefficient of $x_{j_1}^{s_1} \cdots x_{j_k}^{s_k}$ whenever $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$. The set of quasi-symmetric functions is a commutative ring, denoted by QSym, and it is spanned by the monomial quasi-symmetric functions

$$M_{\mathbf{s}} := \sum_{i_1 < \cdots < i_k} x_{i_1}^{s_1} \cdots x_{i_k}^{s_k},$$

$\mathbf{s} = (s_1, \ldots, s_k)$ a composition. Our results are the easier to express using notation introduced by Hoffman [**6**] to study MZVs.

2.1. *Hoffman's algebra*

We consider the non-commutative polynomial algebra

$$\mathfrak{H} := \mathbb{Q}\langle x, y \rangle$$

in two variables, and the two subalgebras

$$\mathfrak{H}^1 := \mathbb{Q} + \mathfrak{H}y, \quad \mathfrak{H}^0 := \mathbb{Q} + x\mathfrak{H}y.$$

A basis of $\mathfrak{H}^1$ consists of words in the non-commuting symbols $z_1, z_2, \ldots$, where $z_n = x^{n-1}y$. A basis for $\mathfrak{H}^0$ consists of those words $z_{s_1} \cdots z_{s_k}$ with $s_1 \geqslant 2$ (along with the empty word 1). There is a $\mathbb{Q}$-linear map $\zeta : \mathfrak{H}^0 \to \mathbb{R}$, taking $z_{s_1} \cdots z_{s_k} \in \mathfrak{H}^0$ to the MZV $\zeta(s_1, \ldots, s_k) \in \mathbb{R}$. The kernel of ζ is identified with the space of relations among MZVs.

The non-commutative ring $\mathfrak{H}^1$ can be given a commutative multiplication $*$, called the *harmonic* (or *stuffle*) product, which is defined recursively. We set

$$1 * \alpha = \alpha * 1 = \alpha$$

for every $\alpha \in \mathfrak{H}^1$. For $k_1, k_2 \geqslant 1$ integers and $\alpha_1, \alpha_2 \in \mathfrak{H}^1$, we set

$$(z_{k_1}\alpha_1) * (z_{k_2}\alpha_2) = z_{k_1}(\alpha_1 * (z_{k_2}\alpha_2)) + z_{k_2}((z_{k_1}\alpha_1) * \alpha_2) + z_{k_1+k_2}(\alpha_1 * \alpha_2).$$

This restricts to give a commutative multiplication on $\mathfrak{H}^0$. The harmonic product is constructed to reflect multiplication of nested sums over integers, and we have

$$\zeta(\alpha_1 * \alpha_2) = \zeta(\alpha_1)\zeta(\alpha_2)$$

for all $\alpha_1, \alpha_2 \in \mathfrak{H}^0$. We write $\mathfrak{H}^1_*$ (respectively, $\mathfrak{H}^0_*$) for the set $\mathfrak{H}^1$ (respectively, $\mathfrak{H}^0$) viewed as a commutative ring with multiplication given by $*$, so that $\zeta : \mathfrak{H}^0_* \to \mathbb{R}$ is a ring homomorphism.

There is an isomorphism of commutative rings $\mathfrak{H}^1_* \cong \mathrm{QSym}$ taking $z_{\mathbf{s}} \in \mathfrak{H}^1$ to the monomial quasi-symmetric function $M_{\mathbf{s}} \in \mathrm{QSym}$. The advantage of using the notation of non-commutative polynomials in x and y is that the MZV $\zeta(s_1, \ldots, s_k)$ can be expressed as an iterated integral, and the specific form of this integral can be read off from the expansion of $z_{s_1} \cdots z_{s_k}$ as a product of the factors x and y. This notation will also be convenient for us in the study of asymptotic relations, though for different reasons.

2.2. *Relations among MZVs*

Problem A is to describe the kernel of $\zeta : \mathfrak{H}^0 \to \mathbb{R}$. There are many known methods to produce elements of $\ker(\zeta)$. We mention three of them here.

(1) Let $\tau : \mathfrak{H} \to \mathfrak{H}$ be the concatenation-reversing automorphism which interchanges x and y. Then τ restricts to an automorphism of $\mathfrak{H}^0$. The Duality Theorem for MZVs, conjectured by Hoffman [**5**] and a consequence of the iterated integral representation, states $\zeta(\alpha) = \zeta(\tau(\alpha))$ for all $\alpha \in \mathfrak{H}^0$, that is, $\tau(\alpha) - \alpha \in \ker(\zeta)$.

(2) Let $D : \mathfrak{H} \to \mathfrak{H}$ be the derivation (for the concatenation product) given on generators by $D(x) = 0$, $D(y) = xy$. By restriction, D gives a derivation of the subalgebra $\mathfrak{H}^0$. Set $\overline{D} := \tau D \tau$ (where τ is the map defined above). Hoffman [**5**, Theorem 5.1; **6**, Theorem 6.3] showed $\overline{D}(\alpha) - D(\alpha) \in \ker(\zeta)$ for all $\alpha \in \mathfrak{H}^0$. This was generalized by Ohno [**10**] and further generalized by Ihara, Kaneko, and Zagier [**8**].

(3) Expanding a product of iterated integrals gives a second way to express a product of MZVs as a linear combination of MZVs, which is in general distinct from the harmonic product. Equality of the two product representations gives the *double shuffle relations* (see [**8**] for a discussion). This can be generalized to allow $\alpha_1, \alpha_2 \in \mathfrak{H}^1$ using a renormalization, leading to the *extended double shuffle relations*, which conjecturally generate $\ker(\zeta)$.

3. *Algebraic framework for classifying weighted congruences and asymptotic relations*

In this section, we introduce a new algebraic framework for studying weighted congruences and asymptotic relations.

3.1. *Completion of* $\mathfrak{H}^1$

As in the case of MZVs, we consider the non-commutative algebra $\mathfrak{H}$ with subalgebra $\mathfrak{H}^1$. The harmonic product $*$ gives a commutative multiplication on $\mathfrak{H}^1$. We set $z_n := x^{n-1}y$ (which has degree $\deg(z_n) = n$), so that a basis of $\mathfrak{H}^1$ consists of the elements

$$z_{\mathbf{s}} := z_{s_1} \cdots z_{s_k},$$

as $\mathbf{s} = (s_1, \ldots, s_k)$ ranges over the compositions. Our framework uses a completion of $\mathfrak{H}^1$.

Definition 3.1. Let $\hat{\mathfrak{H}}^1$ denote the completion of $\mathfrak{H}^1$ with respect to the grading by degree. We may view an element of $\hat{\mathfrak{H}}^1$ as a formal infinite sum

$$\alpha = \sum_{\mathbf{s}} \alpha_{\mathbf{s}} z_{\mathbf{s}}$$

in the non-commuting variables $z_1, z_2, \ldots$, where the summation is taken over all compositions $\mathbf{s} = (s_1, \ldots, s_k)$, and the coefficients $\alpha_{\mathbf{s}}$ are in $\mathbb{Q}$. The elements $z_{\mathbf{s}}$ span a dense subspace of $\hat{\mathfrak{H}}^1$.

For $n \geqslant 0$, we let $\mathbb{I}_n \subset \hat{\mathfrak{H}}^1$ be the set of elements concentrated in degree n and larger:

$$\mathbb{I}_n := \left\{ \sum_{\mathbf{s}} \alpha_{\mathbf{s}} z_{\mathbf{s}} \in \hat{\mathfrak{H}}^1 : \alpha_{\mathbf{s}} = 0 \text{ whenever } w(\mathbf{s}) < n \right\}.$$

These sets are a neighborhood basis of 0 for the topology on $\hat{\mathfrak{H}}^1$. The harmonic product $*$ gives a commutative multiplication on $\hat{\mathfrak{H}}^1$ which is continuous. We write $\hat{\mathfrak{H}}^1_*$ for the set $\hat{\mathfrak{H}}^1$, considered as a commutative topological ring with multiplication given by $*$.

The ring $\hat{\mathfrak{H}}^1_*$ is isomorphic to the completion of the ring of quasi-symmetric functions over $\mathbb{Q}$ with respect to the grading by degree (the isomorphism sends $z_{\mathbf{s}}$ to $M_{\mathbf{s}}$). The open sets $\mathbb{I}_n$ are ideals of $\hat{\mathfrak{H}}^1_*$.

3.2. *The ring of asymptotic numbers*

We next define a ring $\hat{\mathcal{A}}$ that is a kind of p-adic analogue of the ring $\mathcal{A}$ used in the study of finite MZVs. In studying mod p^n congruences, one is led to consider

$$\mathcal{A}_n := \frac{\prod_p \mathbb{Z}/p^n\mathbb{Z}}{\bigoplus_p \mathbb{Z}/p^n\mathbb{Z}}, \quad n \geqslant 1.$$

For each $n \geqslant 1$, there is a surjection $\varphi_n : \mathcal{A}_{n+1} \to \mathcal{A}_n$ coming from reduction modulo p^n. The following definition is new.

Definition 3.2. We define $\hat{\mathcal{A}}$ to be the projective limit of the system of rings $\{\mathcal{A}_n\}$. An element of $\hat{\mathcal{A}}$ may be viewed as an element of the product

$$(r_n) \in \prod_{n=1}^{\infty} \mathcal{A}_n$$

such that $\varphi_n(r_{n+1}) = r_n$ for all $n \geqslant 1$. There are projection maps $\pi_n : \hat{\mathcal{A}} \to \mathcal{A}_n$, sending $(r_n) \in \hat{\mathcal{A}}$ to $r_n \in \mathcal{A}_n$. We put the discrete topology on each $\mathcal{A}_n$ and the projective limit topology on $\hat{\mathcal{A}}$: the sets $\pi_n^{-1}(0) \subset \hat{\mathcal{A}}$ form a neighborhood basis of 0.

As a projective limit of discrete rings, $\hat{\mathcal{A}}$ is complete. It is not locally compact, however, because each basic open subgroup $\pi_n^{-1}(0)$ contains the open subgroup $\pi_{n+1}^{-1}(0)$ of infinite index.

Given an element $a_p \in \mathbb{Z}_p$ for all but finitely many primes p (here $\mathbb{Z}_p$ is the ring of p-adic integers), we get a corresponding element $r_n := (a_p \mod p^n) \in \mathcal{A}_n$ for each n. These elements satisfy $\varphi_n(r_{n+1}) = r_n$ for $n \geqslant 2$, so they determine an element $\hat{\mathcal{A}}$, which we denote $[a_p]$. It can be checked that every element of $\hat{\mathcal{A}}$ arises this way, and that $[a_p] = [b_p]$ if and only if $v_p(a_p - b_p) \to \infty$ as $p \to \infty$ (where v_p is the p-adic valuation).

3.3. *The weighted finite multiple zeta function*

Next we construct a weighted analogue of the finite multiple zeta function. The following definition is new.

Definition 3.3. For each $n \geqslant 1$, define a ring homomorphism

$$\hat{\zeta}_n : \hat{\mathfrak{H}}^1_* \longrightarrow \mathcal{A}_n,$$

$$\sum_{\mathbf{s}} \alpha_{\mathbf{s}} z_{\mathbf{s}} \longmapsto \left[\sum_{w(\mathbf{s})<n} \alpha_{\mathbf{s}} p^{w(\mathbf{s})} H_{p-1}(\mathbf{s}) \mod p^n \right].$$

The maps $\hat{\zeta}_n$ are compatible with the surjections $\varphi_n : \mathcal{A}_{n+1} \to \mathcal{A}_n$, so we define

$$\hat{\zeta} : \hat{\mathfrak{H}}^1_* \longrightarrow \hat{\mathcal{A}}$$

to be the unique map such that $\pi_n \circ \hat{\zeta} = \hat{\zeta}_n$ for all $n \geqslant 1$. Then $\hat{\zeta}$ is called the *weighted finite multiple zeta function*, and for all compositions $\mathbf{s}$,

$$\hat{\zeta}(\mathbf{s}) := \hat{\zeta}(z_{\mathbf{s}}) = [p^{w(\mathbf{s})} H_{p-1}(\mathbf{s})] \in \hat{\mathcal{A}}$$

is called a *weighted finite MZV*.

The map $\hat{\zeta}_n$ is continuous because its kernel contains the open ideal $\mathbb{I}_n \subset \hat{\mathfrak{H}}^1_*$. This implies that $\hat{\zeta}$ is also continuous. By construction, we have

$$\sum_{\mathbf{s}} \alpha_{\mathbf{s}} z_{\mathbf{s}} \in \ker(\hat{\zeta}_n) \iff \sum_{w(\mathbf{s})<n} \alpha_{\mathbf{s}} p^{w(\mathbf{s})} H_{p-1}(\mathbf{s}) \equiv 0 \mod p^n \text{ for large } p,$$

$$\sum_{\mathbf{s}} \alpha_{\mathbf{s}} z_{\mathbf{s}} \in \ker(\hat{\zeta}) \iff \sum_{\mathbf{s}} \alpha_{\mathbf{s}} p^{w(\mathbf{s})} H_{p-1}(\mathbf{s}) \text{ is an asymptotic relation.}$$

Our study of weighted congruences and asymptotic relations amounts to the study of the closed ideals $\ker(\hat{\zeta}_n)$ and $\ker(\hat{\zeta})$. We may view $\ker(\hat{\zeta}_n)/\mathbb{I}_n$ as the space of weighted congruences holding modulo p^n, and $\ker(\hat{\zeta})$ as the space of asymptotic relations. An asymptotic extension of $\alpha \in \ker(\hat{\zeta}_n)$ is then an element $\alpha' \in \ker(\hat{\zeta})$ such that $\alpha' \equiv \alpha \mod \mathbb{I}_n$. We make the following conjecture.

Conjecture A (Asymptotic Extension Conjecture). Every weighted congruence admits an asymptotic extension. Equivalently, there is an equality of ideals:

$$\ker(\hat{\zeta}_n) = \ker(\hat{\zeta}) + \mathbb{I}_n.$$

This conjecture has the following consequence regarding extension of homogeneous relations among finite MZVs: suppose that n is a positive integer and $\alpha_{\mathbf{s}} \in \mathbb{Q}$ for each composition $\mathbf{s}$ of weight n, such that the relation for finite MZVs

$$\sum_{w(\mathbf{s})=n} \alpha_{\mathbf{s}} \zeta_{\mathcal{A}}(\mathbf{s}) = 0$$

holds. Then there are coefficients $\beta_{\mathbf{s}} \in \mathbb{Q}$ for each composition $\mathbf{s}$ of weight $n+1$ such that for all sufficiently large p, we have the weighted congruence

$$\sum_{w(\mathbf{s})=n} \alpha_{\mathbf{s}} H_{p-1}(\mathbf{s}) \equiv p \cdot \sum_{w(\mathbf{s})=n+1} \beta_{\mathbf{s}} H_{p-1}(\mathbf{s}) \mod p^2.$$

In many instances, values of the weighted finite multiple zeta function can be computed using a p-adically convergent series. The following result is not sharp, but will suffice for our needs.

PROPOSITION 3.4. *Let*

$$\alpha = \sum_{\mathbf{s}} \alpha_{\mathbf{s}} z_{\mathbf{s}} \in \hat{\mathfrak{H}}^1,$$

and suppose that there is a positive integer k such that no coefficient $\alpha_{\mathbf{s}}$ has denominator divisible by a kth power bigger than 1. Then the series

$$a_p := \sum_{\mathbf{s}} \alpha_{\mathbf{s}} p^{w(\mathbf{s})} H_{p-1}(\mathbf{s}) \tag{3.1}$$

is p-adically convergent and lies in $\mathbb{Z}_p$ for all sufficiently large p, and $\hat{\zeta}(\alpha) = [a_p]$.

Proof. The hypothesis that the denominators of the $\alpha_{\mathbf{s}}$ are kth power free implies $\alpha_{\mathbf{s}} p^{w(\mathbf{s})} H_{p-1}(\mathbf{s}) \to 0$ p-adically as $w(\mathbf{s}) \to \infty$ for all p, so that the series (3.1) converges to an element of $\mathbb{Q}_p$. We will have $a_p \in \mathbb{Z}_p$ for all p not dividing the denominator of any $\alpha_{\mathbf{s}}$ for $w(\mathbf{s}) < k$. Finally,

$$a_p \equiv \sum_{w(\mathbf{s})<n} \alpha_{\mathbf{s}} p^{w(\mathbf{s})} H_{p-1}(\mathbf{s}) \mod p^n$$

for all p except the finitely many dividing the denominator of some $\alpha_{\mathbf{s}}$ for $w(\mathbf{s}) < n+k$. □

3.4. *Generation of the space of asymptotic relations*

We would like to produce a generating set for the ideal $\ker(\hat{\zeta})$. One difficulty is that from an algebraic viewpoint, this ideal is not very nice: it is likely not even to be countably generated. However, as the kernel of a continuous map, $\ker(\hat{\zeta})$ is closed in $\hat{\mathfrak{H}}^1_*$. As we now show, every closed ideal of $\hat{\mathfrak{H}}^1_*$ is the closure of a particular kind of countably generated ideal.

PROPOSITION 3.5. *Let $I \subset \hat{\mathfrak{H}}^1_*$ be a closed ideal. Then there exists a sequence of elements $\alpha_1, \alpha_2, \ldots \in I$, with $\alpha_n \to 0$, such that I is the closure of the ideal $(\alpha_1, \alpha_2, \ldots)$.*

Proof. We claim that there exists a sequence of finite subsets $S_0, S_1, \ldots \subset I$ such that

$$S_n \subset \mathbb{I}_n \quad \text{and} \quad (S_0) + (S_1) + \cdots + (S_{n-1}) + \mathbb{I}_n = I + \mathbb{I}_n$$

for all $n \geqslant 0$. First we show that the claim implies the desired result. Take the sequence $\alpha_1, \alpha_2, \ldots$ to be the elements of S_0, followed by the elements of S_1, and so on. The condition $S_n \subset \mathbb{I}_n$ implies $\alpha_n \to 0$. By construction, for all $n \geqslant 1$ we have

$$(\alpha_1, \alpha_2, \ldots) + \mathbb{I}_n = I + \mathbb{I}_n. \tag{3.2}$$

The ideals $\mathbb{I}_n$ are a neighborhood basis of 0, so intersecting (3.2) over all n gives

$$\overline{(\alpha_1, \alpha_2, \ldots)} = \overline{I}.$$

The result then follows because I is closed.

To prove the claim, we construct S_n recursively. We start with $S_0 = \emptyset$. Suppose that $S_0, \ldots, S_{n-1}$ have been constructed. The ring $\hat{\mathfrak{H}}^1_*/\mathbb{I}_{n+1}$ is Noetherian (it is in fact finite-dimensional as a vector space), so we can find finitely many elements $\beta_1, \ldots, \beta_k \in I$ whose images in $\hat{\mathfrak{H}}^1_*/\mathbb{I}_{n+1}$ generate the ideal $(I + \mathbb{I}_{n+1})/\mathbb{I}_{n+1}$. By hypothesis $(S_0) + \cdots + (S_{n-1}) + \mathbb{I}_n = I + \mathbb{I}_n$, so we can find $\gamma_1, \ldots, \gamma_k \in \mathbb{I}_n$ so that

$$\beta_j - \gamma_j \in (S_0) + \cdots + (S_{n-1}).$$

One now checks that $S_n := \{\gamma_1, \ldots, \gamma_k\}$ has the desired properties. □

This enables us to formulate a more precise version of Problem D.

Problem D′. Produce a sequence $\alpha_1, \alpha_2, \ldots \in \ker(\hat{\zeta})$, with $\alpha_n \to 0$, such that

$$\ker(\hat{\zeta}) = \overline{(\alpha_1, \alpha_2, \ldots)}.$$

In [**14**], we consider the closed subalgebra $\hat{\Lambda} \subset \hat{\mathfrak{H}}^1_*$ of symmetric functions. We produce a sequence of elements of $\ker(\hat{\zeta}) \cap \hat{\Lambda}$ that, conditionally on a conjecture concerning Bernoulli numbers, generate an ideal with closure $\ker(\hat{\zeta}) \cap \hat{\Lambda}$. This gives a conjectural solution to Problem D′ for the case of symmetric functions. We also give a conditional proof of a suitable modification of Conjecture A, asserting the existence of an asymptotic extension of every *symmetric* weighted congruence.

4. *The asymptotic reversal and duality theorems*

In this section, we prove two results giving asymptotic relations extending congruences modulo p and p^2 in the literature. Both results have the following form: we define a pair Φ and ψ of continuous linear automorphisms of $\hat{\mathfrak{H}}^1$, with Φ the exponential of a topologically nilpotent derivation and $\psi^2 = \mathrm{Id}$. The theorem statements are then that for all $\alpha \in \hat{\mathfrak{H}}^1$,

$$\Phi(\alpha) - \psi(\alpha) \in \ker(\hat{\zeta}).$$

4.1. *The asymptotic reversal theorem*

There is an involution on the set of compositions, taking a composition to its reversal

$$\overline{(s_1, \ldots, s_k)} := (s_k, \ldots s_1).$$

It is known [**7**, Theorem 4.5; **18**, Lemma 3.3] that the homogeneous congruence

$$H_{p-1}(\mathbf{s}) \equiv (-1)^{w(\mathbf{s})} H_{p-1}(\overline{\mathbf{s}}) \pmod p \tag{4.1}$$

holds for all primes p and compositions $\mathbf{s}$. This implies that if $w(\mathbf{s}) = n$, then

$$z_{\mathbf{s}} + (-1)^{n+1} z_{\overline{\mathbf{s}}} \in \ker(\hat{\zeta}_{n+1}).$$

The following theorem provides an asymptotic extension of this congruence.

Theorem 4.1 (Asymptotic Reversal Theorem). *Let $\mathbf{s} = (s_1, \ldots, s_k)$ be a composition. For all primes p, we have a convergent p-adic series equality*

$$H_{p-1}(\mathbf{s}) = (-1)^{w(\mathbf{s})} \sum_{r_1, \ldots, r_k \geqslant 0} \left[\prod_{j=1}^{k} \binom{s_j + r_j - 1}{r_j} \right] p^{r_1 + \cdots + r_k} H_{p-1}(s_k + r_k, \ldots, s_1 + r_1).$$

Reducing this equation modulo p yields (4.1).

Proof. For $1 \leqslant m \leqslant p-1$ an integer, the binomial theorem gives the p-adically convergent identity

$$\frac{1}{(p-m)^s} = \frac{(-1)^s}{m^s}\left(1-\frac{p}{m}\right)^{-s} = (-1)^s \sum_{r \geqslant 0} \binom{s+r-1}{r} \frac{p^r}{m^{s+r}}.$$

We then make the substitutions $n_i \leftrightarrow p - m_i$ in the definition of the multiple harmonic sum, giving

$$\begin{aligned} H_{p-1}(\mathbf{s}) &= \sum_{p-1 \geqslant n_1 > \cdots > n_k \geqslant 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}} = \sum_{p-1 \geqslant m_k > \cdots > m_1 \geqslant 1} \frac{1}{(p-m_1)^{s_1} \cdots (p-m_k)^{s_k}} \\ &= (-1)^{w(\mathbf{s})} \sum_{p-1 \geqslant m_k > \cdots > m_1 \geqslant 1} \sum_{r_1, \ldots, r_k \geqslant 0} \left[\prod_{j=1}^{k} \binom{s_j + r_j - 1}{r} p^{r_j} m_j^{-s_j - r_j} \right] \\ &= (-1)^{w(\mathbf{s})} \sum_{r_1, \ldots, r_k \geqslant 0} \left[\prod_{j=1}^{k} \binom{s_j + r_j - 1}{r_j} \right] p^{r_1 + \cdots + r_k} H_{p-1}(s_k + r_k, \ldots, s_1 + r_1). \end{aligned}$$

□

Using our algebraic framework, we can formulate a version of this theorem in terms of two automorphisms of $\hat{\mathfrak{H}}^1$.

DEFINITION 4.2. Let $\psi_1 : \hat{\mathfrak{H}}^1 \to \hat{\mathfrak{H}}^1$ be the continuous linear map

$$\sum_{\mathbf{s}} \alpha_{\mathbf{s}} z_{\mathbf{s}} \longmapsto \sum_{\mathbf{s}} (-1)^{w(\mathbf{s})} \alpha_{\mathbf{s}} z_{\overline{\mathbf{s}}},$$

which preserves the harmonic product $*$ and reverses the concatenation product. Let $\Phi_1 : \hat{\mathfrak{H}} \to \hat{\mathfrak{H}}$ be the continuous linear map given by

$$\begin{aligned} x &\longmapsto (1-x)^{-1} x = x + x^2 + x^3 + \cdots, \\ y &\longmapsto (1-x)^{-1} y = y + xy + x^2 y + \cdots, \end{aligned}$$

extended by linearity, continuity, and multiplicativity to be a homomorphism for the concatenation product. This restricts to an automorphism of $\hat{\mathfrak{H}}^1$, also called Φ_1.

In terms of ψ_1 and Φ_1, Theorem 4.1 has the following concise form.

THEOREM 4.3 (Asymptotic Reversal Theorem, concise form). *For all* $\alpha \in \hat{\mathfrak{H}}^1$,

$$\Phi_1(\alpha) - \psi_1(\alpha) \in \mathit{ker}(\hat{\zeta}).$$

Proof. By linearity and continuity, it suffices to consider $\alpha = z_{\mathbf{s}}$. We compute

$$\begin{aligned} \Phi_1(z_{\mathbf{s}}) &= \prod_{j=1}^{k} \Phi_1(x^{s_j - 1} y) = \prod_{j=1}^{k} (1-x)^{-s_j} x^{s_j - 1} y \\ &= \prod_{j=1}^{k} \sum_{b_j \geqslant 0} \binom{s_j + b_j - 1}{b_j} x^{s_j - 1 + b} y \\ &= \sum_{b_1, \ldots, b_k \geqslant 0} \left[\prod_{j=1}^{k} \binom{s_j + b_j - 1}{b_j} \right] z_{s_1 + b_1} \cdots z_{s_k + b_k}. \end{aligned}$$

The result now follows from Theorem 4.1. □

It is shown in [8, Proposition 7] that Φ_1 is the exponential of the derivation $\overline{d}$ (for the concatenation product) given on generators by

$$x \longmapsto x^2, \quad y \longmapsto xy.$$

The automorphisms ψ_1 and Φ_1 generate an infinite dihedral group inside the group of continuous linear automorphisms of $\hat{\mathfrak{H}}^1$:

$$\psi_1^2 = (\psi_1\Phi_1)^2 = 1, \quad \#\langle\Phi_1\rangle = \infty.$$

4.2. *The asymptotic duality theorem*

Our next result is also expressed in terms of two automorphisms of $\hat{\mathfrak{H}}^1$.

Definition 4.4. Let $\psi_2 : \hat{\mathfrak{H}} \to \hat{\mathfrak{H}}$ be the continuous linear map which is a ring homomorphism (for the concatenation product) given on generators by

$$x \longmapsto x + y, \quad y \longmapsto -y.$$

This restricts to a linear automorphism of $\hat{\mathfrak{H}}^1 \to \hat{\mathfrak{H}}^1$, which we also denote ψ_2. Let $\Phi_2 : \hat{\mathfrak{H}}^1 \to \hat{\mathfrak{H}}^1$ be the continuous linear automorphism

$$\alpha \longmapsto (1+y)\left(\frac{1}{1+y} * \alpha\right).$$

Our results extend congruences modulo p and p^2 in the literature. To express these congruences, we write $H_{p-1} : \mathfrak{H}^1 \to \mathbb{Q}$ for the linear map given on basis elements by

$$H_{p-1}(z_{\mathbf{s}}) = H_{p-1}(\mathbf{s}).$$

Then ψ_2 restricts to an automorphism of $\mathfrak{H}^1$, and Hoffman [7, Theorem 4.7] shows that for all compositions $\mathbf{s}$, the congruence

$$H_{p-1}(\psi_2(z_{\mathbf{s}})) \equiv H_{p-1}(z_{\mathbf{s}}) \mod p$$

holds for all primes p. This was extended by Zhao [18, Theorem 2.11], who proves a result equivalent to the congruence

$$H_{p-1}(\psi_2(z_{\mathbf{s}})) \equiv H_{p-1}(z_{\mathbf{s}}) + pH_{p-1}(1 \sqcup \mathbf{s}) \mod p^2,$$

where $1 \sqcup (s_1, \ldots, s_k)$ is the composition $(1, s_1, \ldots, s_k)$.

Theorem 4.5 (Asymptotic Duality Theorem). *For all $\alpha \in \hat{\mathfrak{H}}^1$, we have*

$$\Phi_2(\alpha) - \psi_2(\alpha) \in \ker(\hat{\zeta}).$$

Proof. By linearity, it suffices to consider $\alpha = z_{\mathbf{s}}$. First we show that, for n a fixed non-negative integer and p a prime, $\binom{p}{n}$ can be expressed as a sum involving weighted multiple harmonic sums. We have

$$\begin{aligned}\binom{p}{n} &= \frac{p(p-1)\cdots(p-n+1)}{n!} \\ &= (-1)^{n+1}\frac{p}{n}\left(1-\frac{p}{1}\right)\left(1-\frac{p}{2}\right)\cdots\left(1-\frac{p}{n-1}\right) \\ &= (-1)^{n+1}\frac{p}{n}\sum_{j\geqslant 0}(-1)^j p^j H_{n-1}(\{1\}^j).\end{aligned}$$

A result of Hoffman [7, Theorem 4.2 and the proof of Theorem 4.6] shows that for any composition $\mathbf{s}$,

$$H_{p-1}(\psi_2(z_{\mathbf{s}})) = \sum_{n=1}^{p}(-1)^{n+1}\binom{p}{n} H_{p-1}(\mathbf{s}).$$

Multiplying through by $p^{w(\mathbf{s})}$ and using our expression for $\binom{p}{n}$, we have

$$\begin{aligned}\hat{\zeta}(\psi_2(z_{\mathbf{s}})) &= [p^{w(\mathbf{s}^*)}H_{p-1}(\psi_2(z_{\mathbf{s}}))] = \left[\sum_{n=1}^{p}\binom{p}{n}(-1)^{n+1}p^{w(\mathbf{s})}H_{n-1}(\mathbf{s})\right] \\ &= \left[p^{w(\mathbf{s})}H_{p-1}(\mathbf{s}) + \sum_{n=1}^{p-1}\frac{p}{n}p^{w(\mathbf{s})}\left(\sum_{j\geqslant 0}(-1)^j p^j H_{n-1}(\{1\}^j)\right)H_{n-1}(\mathbf{s})\right] \\ &= \hat{\zeta}\left(z_{\mathbf{s}} + y\left(\frac{1}{1+y} * z_{\mathbf{s}}\right)\right).\end{aligned}$$

The above computation is linear and continuous in $z_{\mathbf{s}}$, so for all $\alpha \in \hat{\mathfrak{H}}^1$ we have

$$\alpha + y\left(\frac{1}{1+y} * \alpha\right) - \psi_2(\alpha) \in \ker(\hat{\zeta}). \tag{4.2}$$

To obtain the desired result, we observe the asymptotic relation

$$\sum_{n\geqslant 1}(-1)^{n+1}p^n H_{p-1}(\underbrace{1,\ldots,1}_{n}) = 0, \tag{4.3}$$

which follows for $p \geqslant 3$ from [12, Proposition 2.1 (with $n = p-1$, $j = 0$)]. This implies $y - y^2 + y^3 - \cdots = (y/(1+y)) \in \ker(\hat{\zeta})$. Since $\ker(\hat{\zeta})$ is an ideal of $\hat{\mathfrak{H}}^1_*$, (4.2) and (4.3) together imply that the element

$$-\frac{y}{1+y} * \alpha + \alpha + y\left(\frac{1}{1+y} * \alpha\right) - \psi_2(\alpha) = \Phi_2(\alpha) - \psi_2(\alpha)$$

is in $\ker(\hat{\zeta})$. □

It is shown in [8, Proposition 6] that Φ_2 is the exponential of the derivation (for the concatenation product) given on generators by

$$x \longmapsto 0, \quad y \longmapsto -\sum_{n\geqslant 1}\frac{x^n y + y x^{n-1} y}{n},$$

so in particular Φ_2 preserves the concatenation product. We also have $\psi_2^2 = \mathrm{Id}$.

5. *Calculations in low weight*

In the previous section, we gave two methods to produce elements of $\ker(\hat{\zeta})$. In this section, we show how to use these elements to write down various weighted congruences. The techniques can be adapted to accommodate additional asymptotic relations that may become known in the future. A script for performing these and other computations in computer algebra system Mathematica can be found in [13].

5.1. *Numerical computation*

For $i = 1, 2$, we define functions $f_i : \hat{\mathfrak{H}}^1 \to \hat{\mathfrak{H}}^1$ by

$$f_i(\alpha) := \Phi_i(\alpha) - \psi_i(\alpha),$$

where Φ_i, ψ_i are defined in Section 4. Each f_i is $\mathbb{Q}$-linear, maps each $\mathbb{I}_n$ into itself, and the results of Section 4 imply $f_i(\alpha) \in \ker(\hat{\zeta})$ for all $\alpha \in \hat{\mathfrak{H}}^1$.

We will work in the quotient $\hat{\mathfrak{H}}^1/\mathbb{I}_n$, which is finite-dimensional. For computational purposes, we must choose an ordered basis of $\hat{\mathfrak{H}}^1/\mathbb{I}_n$, so we choose the elements $z_{\mathbf{s}}$ as $\mathbf{s}$ ranges over the compositions of weight less than n. We find it convenient to order the compositions determining our basis elements first by weight (lowest weight comes first), then by lexicographic order (smaller numbers come first).

We have

$$\alpha * f_i(\beta) \in \ker(\hat{\zeta}) \tag{5.1}$$

for $i = 1, 2$ and all $\alpha, \beta \in \hat{\mathfrak{H}}^1$. Because we work in $\hat{\mathfrak{H}}^1/\mathbb{I}_n$ and (5.1) is bilinear in α and β, it suffices to take α, β in our chosen basis for $\hat{\mathfrak{H}}^1/\mathbb{I}_n$. This gives us a finite collection of elements.

Definition 5.1. Let M_n be the matrix whose rows are indexed by triples $(\mathbf{s}_1, \mathbf{s}_2, i)$ with $w(\mathbf{s}_1) + w(\mathbf{s}_2) < n$ and $i \in \{1, 2\}$, whose columns are indexed by compositions $\mathbf{s}$ with $w(\mathbf{s}) < n$, and whose entry in row $(\mathbf{s}_1, \mathbf{s}_2, i)$, column $\mathbf{s}$ is the coefficient of $z_{\mathbf{s}}$ in

$$z_{\mathbf{s}_1} * f_i(z_{\mathbf{s}_2}) \in \hat{\mathfrak{H}}^1/\mathbb{I}_n.$$

The entries of M_n are rational numbers, and we identify the row span of M_n with the space weighted congruence holding modulo p^n which can be derived from the results of Section 4. In particular, all such weighted congruences admit asymptotic extensions. We say that a compositions $\mathbf{s}$ is *essential* if $\mathbf{s}$ is not a pivot column of M_n for some (or equivalently all) $n > w(\mathbf{s})$.

This notion of essential depends on our choice of ordering for compositions. It also depends on the source of asymptotic relations (here we consider only those asymptotic relations coming from the results of Section 4). The set

$$\{\hat{\zeta}_n(z_{\mathbf{s}}) : w(\mathbf{s}) < n,\ \mathbf{s} \text{ essential}\}$$

spans the image of $\hat{\zeta}_n$. For example, when $n = 7$, we find the essential compositions are $(2, 1)$, $(4, 1)$, and $(4, 1, 1)$.

5.2. *Example computations*

Given an element $\alpha \in \hat{\mathfrak{H}}^1$ and an integer $n \geqslant 1$, one may compute the matrix M_n described above and perform row reduction to find a unique element

$$\varphi(\alpha) = \sum_{\substack{w(\mathbf{s})<n \\ \mathbf{s} \text{ essential}}} \alpha'_{\mathbf{s}} z_{\mathbf{s}},$$

which is congruent modulo the row span of M_n to the image of α in $\hat{\mathfrak{H}}^1/\mathbb{I}_n$. This will mean $\alpha - \varphi(\alpha) \in \ker(\hat{\zeta}) + \mathbb{I}_n \subset \ker(\hat{\zeta}_n)$. The map $\alpha \mapsto \varphi(\alpha)$ is linear. Because we have ordered compositions by weight, φ takes $\mathbb{I}_m$ to $\mathbb{I}_m$ for all $m \geqslant 1$.

5.2.1. *Extending known congruences.* Given $\alpha \in \hat{\mathfrak{H}}$, we have $\alpha - \varphi(\alpha) \in \ker(\hat{\zeta}) + \mathbb{I}_n$, so we get a weighted congruence admitting an asymptotic extension. As an example, we take $\alpha = z_{(1)}$, $n = 7$. Numerical computation gives

$$\varphi(\alpha) = -\tfrac{1}{3} z_{(2,1)} + \tfrac{1}{6} z_{(4,1)} + \tfrac{1}{9} z_{(4,1,1)},$$

so the weighted congruence

$$H_{p-1}(1) + \tfrac{1}{3}p^2 H_{p-1}(2,1) - \tfrac{1}{6}p^4 H_{p-1}(4,1) - \tfrac{1}{9}p^5 H_{p-1}(4,1,1) \equiv 0 \mod p^6$$

holds for all sufficiently large primes p, and this congruence admits an asymptotic extension. Reducing modulo p^5 gives the congruence (1.3), which is an extension of Wolstenholme's congruence. This calculation can be repeated with α replaced by any element of $\hat{\mathfrak{H}}^1$.

5.2.2. *Congruences holding modulo high powers.* Given $\alpha_1, \ldots, \alpha_k \in \hat{\mathfrak{H}}^1$ and $n \geqslant 1$, we can compute $\varphi(\alpha_1), \ldots, \varphi(\alpha_n)$, which lie in a vector space spanned by the essential compositions of weight less than n. This space is not too big, and we may attempt to find a non-trivial linear relation

$$\sum_{i=1}^{k} a_i \varphi(\alpha_i) = 0 \in \hat{\mathfrak{H}}^1 / \mathbb{I}_n.$$

If we can find such a relation, then we will have

$$\sum_{i=1}^{k} a_i \alpha_i \in \ker(\hat{\zeta}) + \mathbb{I}_n$$

giving a weighted congruence having an asymptotic extension.

As an example, we set $n = 10$ and take the α_i to be $z_{\mathbf{s}}$ for compositions $\mathbf{s}$ of weight 5 and depth at most 2: $\alpha_1 = z_{(1,4)}$, $\alpha_2 = z_{(2,3)}$, $\alpha_3 = z_{(3,2)}$, $\alpha_4 = z_{(4,1)}$, $\alpha_5 = z_{(5)}$. We compute the corresponding elements $\varphi(\alpha_1), \ldots, \varphi(\alpha_5)$, and we find that they satisfy the linear relation

$$3\varphi(\alpha_1) - \varphi(\alpha_2) - \varphi(\alpha_3) + 3\varphi(\alpha_4) + 2\varphi(\alpha_5) = 0.$$

This shows that for p sufficiently large, we have the homogeneous weighted congruence

$$3H_{p-1}(1,4) - H_{p-1}(2,3) - H_{p-1}(3,2) + 3H_{p-1}(4,1) + 2H_{p-1}(5) \equiv 0 \mod p^5,$$

and that this congruence admits an asymptotic extension.

6. *The image of* $\hat{\zeta}$

In this section, we investigate the image of the weighted finite multiple zeta function $\hat{\zeta}$. We start with the following proposition.

PROPOSITION 6.1. *The image of $\hat{\zeta}$ is closed in $\hat{\mathcal{A}}$.*

This is a consequence of a more general result. A complete topological ring is said to be *linearly topologized* if the topology admits a neighborhood basis of ideals. Both topological rings $\hat{\mathfrak{H}}^1_*$ and $\hat{\mathcal{A}}$ considered so far are linearly topologized. Proposition 6.1 is an immediate consequence of a more general fact.

LEMMA 6.2. *Let $\varphi : R_1 \to R_2$ be a continuous homomorphism of linearly topologized rings, and suppose that R_1 is the projective limit of a (countable) sequence of Artinian rings. Then the image of φ is closed.*

Proof. Let $R_1 \supset J_1 \supset J_2 \supset \cdots$ and $\{I_\alpha \subset R_2\}$ be ideals of forming neighborhood bases of 0, with R_1/J_i Artinian for each i. Then for every α,

$$J_\alpha := \varphi^{-1}(I_\alpha)$$

is an open ideal of R_1, so contains some I_n, say $I_{n(\alpha)}$. We get maps

$$\varphi_\alpha : R_1/I_{n(\alpha)} \longrightarrow R_2/I_\alpha.$$

Suppose that $r \in R_2$ is in the closure of the image of φ. For each α, define

$$S_\alpha := \varphi_\alpha^{-1}(r + I_\alpha) \subset R_1.$$

This is a coset of the ideal $I_{n(\alpha)}$. The image of φ intersects $r + I_\alpha$ non-trivially, so S_α is non-empty. For $m > n$, the quotient map $R_1/I_m \to R_1/I_n$ takes S_m into S_n. The image of S_m is a coset of an ideal in R_1/I_n, and by hypothesis the ring R_1/I_n is Artinian so satisfies the descending chain condition on cosets of ideals. This means the inverse system $S_1 \leftarrow S_2 \leftarrow \cdots$ satisfies the Mittag–Leffler Condition, so since each S_n is non-empty, there is an element r' of R_1 that maps into every S_n. This r' will map to r by φ. □

6.1. *Asymptotic representability*

The closed subring $\operatorname{im}(\hat{\zeta}) \subset \hat{\mathcal{A}}$ is very small, in the following sense: for each integer $n \geqslant 1$, the map $\hat{\zeta}_n : \hat{\mathfrak{H}}^1_* $ to $\mathcal{A}_n$ factors through the countable ring $\hat{\mathfrak{H}}^1_*/I_n$. This means the $\pi_n(\operatorname{im}(\hat{\zeta})) \subset \mathcal{A}_n$ is countable. However, $\mathcal{A}_n$ is the quotient of a countably infinite product of finite sets by a countable ideal, so has continuum cardinality. Nevertheless, many interesting quantities appearing in arithmetic are in the image of $\hat{\zeta}$, and we produce some below. First we give a definition.

Definition 6.3. A collection of elements $a_p \in \mathbb{Z}_p$ is said to be *asymptotically representable by weighted multiple harmonic sums*, or simply *asymptotically representable*, if the corresponding element $[a_p] \in \hat{\mathcal{A}}$ is in the image of $\hat{\zeta}$. In other words, the collection $a_p \in \mathbb{Z}_p$ is asymptotically representable if and only if there are coefficients $\alpha_{\mathbf{s}} \in \mathbb{Q}$ not depending on p such that for all $n \geqslant 1$, the congruence

$$a_p \equiv \sum_{w(\mathbf{s})<n} \alpha_{\mathbf{s}} p^{w(\mathbf{s})} H_{p-1}(\mathbf{s}) \mod p^n$$

holds for all sufficiently large p.

Our next result concerns a family of sums related to multiple harmonic sums.

Theorem 6.4. *For all compositions* $\mathbf{s} = (s_1, \ldots, s_k)$ *and all positive integers* r, *the generalized weighted multiple harmonic sum*

$$p^{w(\mathbf{s})} H^{(p)}_{pr}(\mathbf{s}) := p^{w(\mathbf{s})} \sum_{\substack{pr \geqslant n_1 > \cdots > n_k \geqslant 1 \\ p \nmid n_1 n_2 \cdots n_k}} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}$$

is asymptotically representable.

Proof. First we note the convergent p-adic expansion

$$(jp + n)^{-s} = \sum_{i=0}^{\infty} \binom{-s}{i} j^i \frac{p^i}{n^{s+i}},$$

which holds for $j \in \mathbb{Z}_p$, $n \in \mathbb{Z}_p^\times$, $s \in \mathbb{Z}$. Writing $n_i = j_i p + m_i$,

$$\begin{aligned}
p^{w(\mathbf{s})} H_{pr}^{(p)}(\mathbf{s}) &= \sum_{j_1 \geqslant \cdots \geqslant j_k = 0}^{r-1} \left(\sum_{\substack{p-1 \geqslant m_1, \ldots, m_k \geqslant 1 \\ m_i > m_{i+1} \text{ if } j_i = j_{i+1}}} \frac{p^{w(\mathbf{s})}}{(j_1 p + m_1)^{s_1} \cdots (j_k p + m_k)^{s_k}} \right) \\
&= \sum_{j_1 \geqslant \cdots \geqslant j_k = 0}^{r-1} \left(\sum_{\substack{p-1 \geqslant m_1, \ldots, m_k \geqslant 1 \\ m_i > m_{i+1} \text{ if } j_i = j_{i+1}}} \prod_{q=1}^{k} \sum_{i_q=0}^{\infty} \binom{-s_q}{i_q} j_q^{i_q} \frac{p^{s_q + i_q}}{m_q^{s_q + i_q}} \right) \\
&= \sum_{\substack{0 \leqslant j_1, \ldots, j_k \leqslant r-1 \\ 0 \leqslant i_1, \ldots, i_k}} \left(\prod_{q=1}^{k} \binom{-s_q}{i_q} j_q^{i_q} \right) \prod_{j=0}^{r-1} p^{w(\mathbf{t}_j)} H_{p-1}(\mathbf{t}_j),
\end{aligned}$$

where

$$\mathbf{t}_j := (s_a + i_a, \ldots, s_b + i_b), \quad \{a, a+1, \ldots, b\} = \{n : j_n = j\}.$$

All of the coefficients are integers, so Proposition 3.4 implies

$$[p^{w(\mathbf{s})} H_{pr}^{(p)}(\mathbf{s})] = \hat{\zeta} \left(\sum_{\substack{0 \leqslant j_1, \ldots, j_k \leqslant r-1 \\ 0 \leqslant i_1, \ldots, i_k}} \left(\prod_{q=1}^{k} \binom{-s_q}{i_q} j_q^{i_q} \right) z_{\mathbf{t}_0} * \cdots * z_{\mathbf{t}_{r-1}} \right). \qquad \square$$

Our next result concerns binomial coefficients of a certain shape.

THEOREM 6.5. *For all integers k, r, with $r \geqslant 0$, the binomial coefficient*

$$\binom{kp}{rp}$$

is asymptotically representable.

Proof. For p odd, we write

$$\begin{aligned}
\binom{kp}{rp} &= \frac{(kp)(kp-1)\cdots((k-r)p+1)}{(rp)(rp-1)\cdots(1)} \\
&= \binom{k}{r} \prod_{\substack{j=1 \\ p \nmid j}}^{rp-1} \left(1 - \frac{kp}{j}\right) = \binom{k}{r} \sum_{n \geqslant 0} (-k)^n p^n H_{rp}^{(p)}(\{1\}^n).
\end{aligned}$$

The result now follows from the asymptotic representability of $p^n H_{rp}^{(p)}(\{1\}^n)$ and the fact that $\operatorname{im}(\hat{\zeta})$ is closed. $\square$

Our final result gives an asymptotic representation of values of the p-adic zeta function, which we now describe. Values taken by the Riemann zeta function at negative integers are rational, and can be expressed in terms of Bernoulli numbers. More generally, for any Dirichlet character χ, values of the L-function $L(s, \chi)$ at negative integers lie in the field generated over $\mathbb{Q}$ by the values of χ, so in particular are algebraic. These values can be expressed in terms of generalized Bernoulli numbers.

Let p be an odd prime, $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$, and $\mathbb{C}_p$ be the completion of an algebraic closure of $\mathbb{Q}_p$ (the field of p-adic numbers). We fix once and for all embeddings of $\overline{\mathbb{Q}}$ into $\mathbb{C}$ and into $\mathbb{C}_p$. This allows us to identify algebraic elements of $\mathbb{C}$ as living in $\mathbb{C}_p$. Kummer's congruences for the generalized Bernoulli numbers imply that for any primitive

Dirichlet character χ, the function

$$\left(1-\chi(p)p^{-s}\right) L(s,\chi)$$

is p-adically continuous when restricted to the negative integers s in a fixed residue class mod $p-1$. For $\chi \neq 1$, the p-adic L-function of Kubota–Leopoldt is the unique continuous function $\mathbb{Z}_p \to \mathbb{C}_p$, $s \mapsto L_p(s,\chi)$, agreeing with $(1-\chi(p)p^{-s})L(s,\chi)$ when s is a negative integer congruent to 1 mod $p-1$. For $\chi = 1$, $L_p(s,\chi)$ is continuous except for a simple pole at $s=1$.

The Teichmüller character $\omega : (\mathbb{Z}/(p))^{\times} \to \mathbb{Q}_p^{\times}$ is the unique group homomorphism such that $\omega_p(n) \equiv n \mod p$ for all $n \in \mathbb{Z}$ not divisible by p. The p-adic L-function $L_p(s,\omega_p^{1-k})$ agrees with $(1-\chi(p)p^{-s})L(s,\chi)$ for negative integers s congruent to k mod $p-1$. For $k \geqslant 2$, the p-adic zeta value is defined by

$$\zeta_p(k) := L_p(k,\omega^{1-k}) \in \mathbb{Q}_p.$$

It is worth noting that $\zeta_p(k)$ is not p-adically continuous as a function of k, but comes from $p-1$ different continuous functions, one defined on each residue class mod $p-1$. The vanishing of the odd Bernoulli numbers implies $\zeta_p(2k)=0$ for $k \geqslant 1$.

THEOREM 6.6. *For all integers $k \geqslant 2$, $p^k\zeta_p(k)$ is asymptotically representable.*

Proof. In [**17**, Theorem 1 (with $n=1$)], Washington shows that for $r \geqslant 1$, we have a convergent p-adic series identity for a multiple harmonic sums in terms of p-adic ζ-values:

$$\begin{aligned} H_{p-1}(s) &= -\sum_{k=1}^{\infty} \binom{-s}{k} p^k \zeta_p(s+k) \\ &= \sum_{k=s+1}^{\infty} (-1)^{k+s+1} \binom{k-1}{s-1} p^{k-s}\zeta_p(k). \end{aligned}$$

Fix an integer $n \geqslant 2$. Using the identity above, we can compute

$$\begin{aligned} &\sum_{s\geqslant n-1} \frac{(-1)^{s+n+1}}{n-1} \binom{s-1}{n-2} B_{s+1-n} p^s H_{p-1}(s) \\ &= \sum_{s\geqslant n-1} \frac{(-1)^{s+n+1}}{n-1} \binom{s-1}{n-2} B_{s+1-n} p^s \sum_{k=s+1}^{\infty} (-1)^{k+s+1} \binom{k-1}{s-1} p^{k-s}\zeta_p(k) \\ &= \sum_{k=n}^{\infty} \frac{(-1)^{k+n}}{n-1} p^k \zeta_p(k) \sum_{s=n-1}^{k-1} \binom{k-1}{s-1}\binom{s-1}{n-2} B_{s+1-n} \\ &= \sum_{k=n}^{\infty} \frac{(-1)^{k+n}}{n-1} \binom{k-1}{n-2} p^k \zeta_p(k) \sum_{s=n-1}^{k-1} \binom{k+1-n}{s+1-n} B_{s+1-n} \\ &= \sum_{k=n}^{\infty} \frac{(-1)^{k+n}}{n-1} \binom{k-1}{n-2} p^k \zeta_p(k) \sum_{s=0}^{k-n} \binom{k+1-n}{s} B_s. \end{aligned}$$

It is known that

$$\sum_{s=0}^{k-n} \binom{k+1-n}{s} B_s = \begin{cases} 1 & \text{if } k=n, \\ 0 & \text{otherwise} \end{cases}$$

(see [2, p. 7]), so we obtain

$$\sum_{s\geqslant n-1} \frac{(-1)^{s+n+1}}{n-1}\binom{s-1}{n-2} B_{s+1-n} p^s H_{p-1}(s) = p^n \zeta_p(n).$$

The von Staudt–Clausen Theorem implies that B_k has a square-free denominator, and the factor of $n-1$ in the denominator is constant, so Proposition 3.4 gives

$$[p^n \zeta_p(n)] = \hat{\zeta}\left(\sum_{s\geqslant n-1} \frac{(-1)^{s+n+1}}{n-1}\binom{s-1}{n-2} B_{s+1-n} z_s\right). \qquad \square$$

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